A Nonmonotonic Conditional Logic for Belief Revision

Part 1: Semantics and Logic of Simple Conditionals

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Abstract

Using Gärdenfors's notion of epistemic entrenchment, we develop the semantics of a logic which accounts for the following points. It explains why we may generally infer If $\neg$A then B if all we know is A$\vee$B while must not generally infer If $\neg$A then B if all we know is $\{$A$\vee$B,A$\}$. More generally, it explains the nonmonotonic nature of the consequence relation governing languages which contain conditionals, and it explains how we can deduce conditionals from premise sets without conditionals. Depending on the language at hand, our logic provides different ways of keeping the Ramsey test and getting round the Gärdenfors triviality theorem. We indicate that consistent additions of new items of belief are not to be performed by transitions to logical expansions.

1 Introduction

1.1 An example

Imagine that you are walking along a long and lonely beach. It is a beautiful night. Still you feel somewhat uncomfortable. You are hungry. But you know that at the end of the beach there are two restaurants, one of them run

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Published in *The Logic of Theory Change*,
eds. André Fuhrmann and Michael Morreau,
Springer LNAI 465, Berlin 1991, pp. 135–181.

by Annie, the other one by Ben. There are no other buildings around. Now you are still far away from the restaurants, but you happen to perceive a shimmering light there, without being able to make out whether it comes from Annie's or Ben's restaurant. So you form the belief that either Annie's or Ben's restaurant is open. And also, you are willing to accept the conditional

> If Annie's restaurant is not open (then) Ben's restaurant will be open. (1)

Approaching the promising end of the beach, you see that Annie's restaurant is lit while Ben's is unlit. You form the new beliefs that Annie's but not Ben's restaurant is open. You have just learned something new, nothing causes any contradiction. But surprisingly you have lost the conditional (1). You no longer believe that if Annie's restaurant is not open then Ben's restaurant will be open, nor do you assent to the (more appropriate) subjunctive variant

> If Annie's restaurant were not open (then) Ben's restaurant would be open. (2)

Put in more formal terms, the premise of your belief state in the first situation may be taken to be $A \vee B$. Later on you add new pieces of information, viz., A and $\neg B$. Representing the natural language conditional 'if ... then ... ' by the formal connective $\Box\!\!\rightarrow$, we find that you can infer $\neg A \Box\!\!\rightarrow B$ at the outset of your beach walk, but that you cannot infer $\neg A \Box\!\!\rightarrow B$ after spotting the light source in Annie's restaurant:

$$\neg A \Box\!\!\rightarrow B \in \mathrm{Cn}(\{A \vee B\}) \text{ , but}$$
$$\neg A \Box\!\!\rightarrow B \notin \mathrm{Cn}(\{A \vee B, A, \neg B\})$$
$$(\text{or } \neg A \Box\!\!\rightarrow B \notin \mathrm{Cn}(\{A \vee B, A \& \neg B\})).$$

Conditionals thus exhibit a *non-monotonic* behaviour. That is, in the context of a language which contains conditionals, we cannot expect to have a plausible consequence relation Cn such that $\Gamma \subseteq \Gamma'$ automatically implies $\mathrm{Cn}(\Gamma) \subseteq \mathrm{Cn}(\Gamma')$. This is my first point. My second one is that Cn should include some kind of *conditional logic*. In the initial situation, it appears quite correct to infer the natural language conditional (1) (not just the material conditional $\neg A \rightarrow B$!) from the premises in which no conditional connective occurs.

Notice that it seems very natural to switch from considering an indicative to considering the corresponding subjunctive conditional in this example.[1] We will in fact presuppose in this paper that, roughly, both types of conditionals are susceptible to a unified account employing the so-called *Ramsey test*:[2]

(R) $A \Box\!\!\rightarrow B$ is accepted in a belief state if and only if updating this belief state so as to accomodate A leads to a belief state where B is accepted.

1.2 The role of consequence relations in Gärdenfors's incompatibility theorem

The points just made in the intuitive example have a counterpart in a meanwhile notorious abstract result. Gärdenfors (1986; see Gärdenfors 1988, Sections 7.4-7.7) has shown that the Ramsey test is incompatible with a small number of apparently innocuous and reasonable requirements for updating belief states. The most important one is the *preservation principle*:

(P) If a sentence A is consistent with a belief state then updating this belief state so as to add A leads to a belief state which includes all sentences accepted in the original belief state.[3]

Leaving aside technical niceties, all proofs for the Gärdenfors incompatibility theorem that can be found in the literature run like this. Start with a belief

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[1]The example is a variation of an example to be found in Hansson (1989). The crucial difference from Hansson's hamburger example is that in my case spotting the light in Annie's restaurant completely overrides the earlier piece of information that Annie's *or* Ben's restaurant is lit. In this way my example is also meant to refute the suggestion of Morreau (1990) that the evaluation of conditionals always depends on the order of incoming information. Morreau's analysis predicts, wrongly I believe, that conditionals cannot be lost after consistent updates of belief states. See Rott (1990).

[2]Since Adams published his famous Kennedy example, most writers have refrained from venturing a unified analysis of indicative and subjunctive conditionals. I think, however, that the principle of compositionality should be applied here. If there are differences in meaning between indicative and subjunctive conditionals, they should be attributed to the different grammatical moods and/or tenses rather than to the connective 'if' itself.

[3](R) and (P) could be weakened by requiring that A and B be "objective sentences", i.e., non-conditionals. This would not make a difference for the following. However, while I reject (P), I shall accept a modified form of the preservation principle saying that *objective* sentences are preserved under consistent updates.

state K that is totally ignorant with respect to two sentences A and B. Let K′ and K″ be the belief states that are obtained after adding A and B to K, respectively. Now the preservation principle says that adding ¬(A&B) to K′ and K″ will not throw out A and B from K′ and K″, respectively. Applying the Ramsey test, this gives that ¬(A&B)□→A is in K′ and ¬(A&B)□→B is in K″. Now consider K‴ which is the resulting belief state after adding A and B (or after adding A&B) to K. It is usually stipulated or just taken for granted that K‴ is a superset of both K′ and K″. Hence both ¬(A&B)□→A and ¬(A&B)□→B are in K‴, hence, by another application of the Ramsey test, A and B are in the update of K‴ which is necessary in order to accomodate ¬(A&B). But of course, ¬(A&B) should be in this update as well. So this update is inconsistent, in contradiction to a quite modest principle of consistency maintenance.

The reader will already have guessed the point where I do not agree. It is the stipulation that K‴ be a superset of K′ and K″. Actually, most writers identify consistent additions of beliefs with logical *expansions*:

$$\begin{gathered} \mathrm{K}' = \mathrm{Cn}(\mathrm{K}\cup\{\mathrm{A}\}) \\ \mathrm{K}'' = \mathrm{Cn}(\mathrm{K}\cup\{\mathrm{B}\}) \\ \mathrm{K}''' = \mathrm{Cn}(\mathrm{K}\cup\{\mathrm{A},\mathrm{B}\}) \\ (\text{or } \mathrm{K}''' = \mathrm{Cn}(\mathrm{K}\cup\{\mathrm{A}\&\mathrm{B}\})). \end{gathered}$$

It follows that K‴ is a superset of both K′ and K″, if one can presuppose that Cn is monotonic, or respectively, if it satisfies the similar, slightly weaker principle of *classical monotonicity*: if $\mathrm{Cn}_0(\Gamma)\subseteq\mathrm{Cn}_0(\Gamma')$ then $\mathrm{Cn}(\Gamma)\subseteq\mathrm{Cn}(\Gamma')$. But we saw in the introductory example that neither monotonicity nor classical monotonicity is warranted in languages containing conditionals.

Another way to make precise the intuitive idea behind the proofs of the Gärdenfors incompatibility theorem is to keep the expansion idea for K′ and K″ but to identify K‴ with Cn(K′∪{B}) and Cn(K″∪{A}). This would guarantee that K‴ is a superset of K′ and K″; but it of course assumes that Cn(K′∪{B}) and Cn(K″∪{A}) are the same set. We shall see below, however, that this identification is not valid either in our modelling of Cn, as long as we are concerned with the language L_1 specified below. In the more comprehensive language L_2, Cn(K′∪{B}) and Cn(K″∪{A}) will be identical, but only at the expense of inconsistency.[4]

[4]In anticipation of things to be explained below: In L_1, Cn(Cn({A})∪{B}) corresponds to the sentences satisfied by the E-relation based on $\bot\prec\mathrm{B}\prec\mathrm{A}\prec\top$, Cn(Cn({B})∪{A})

It is, however, utterly implausible to assume that adding B after A (or adding A after B) to a belief state that is totally ignorant about A and B leads to an inconsistent belief state. In Rott (1989a) I argued that the right lesson to be drawn from the Gärdenfors incompatibility result is that *consistent revisions* by new items of belief, which I call *additions*, are not to be identified with expansions. Now let us write $K^{o}{}_{A}$ for the result of adding A to K. In the final analysis, we see how the puzzle caused by the Gärdenfors incompatibility theorem gets resolved. We will develop an account of how consistent additions of sentences are possible by adding new pieces of information to some set of premises from which a belief state is generated. What we then get is that $(K^{o}{}_{A})^{o}{}_{B}$ equals $(K^{o}{}_{B})^{o}{}_{A}$ but that no longer $(K^{o}{}_{A}) \subseteq (K^{o}{}_{A})^{o}{}_{B}$ or $(K^{o}{}_{B}) \subseteq (K^{o}{}_{B})^{o}{}_{A}$. We summarize our preliminary overview of the different possiblities of cutting the chain of proof of the Gärdenfors incompatibility theorem in the table on the next page.[5]

1.3 Program

This paper is intended to be the first part of a trilogy. We shall base the notion of a belief revision model on the concept of a relation of epistemic entrenchment ("E-relation"). We discuss the properties, the motivation and the finite representability of E-relations. Then we say what it means that a relation of epistemic entrenchment satisfies a sentence. Sentences of four different languages will be considered.[6] First, we have the purely "truth-functional" language L_0 of propositional logic with the symbols $\neg$, &, $\vee$, $\rightarrow$, $\perp$ and $\top$. In the present paper we will then examine the language L_1 with an additional binary conditional operator $\Box\!\!\rightarrow$ which connects sentences from L_0. In the second part, we extend L_1 to L_2 by admitting the possibility that L_1-sentences are connected by the classical operators of L_0; in particular L_1 allows for negations and disjunctions of conditionals. Finally, in the third part of the trilogy, we shall make some comments on L_3 which extends L_2 by permitting nested conditionals. This last part will largely be devoted

corresponds to $\perp \prec A \prec B \prec \top$. In L_2, however, B cannot be consistently added to $Cn(\{A\})$ at all.

[5]Admittedly, it is unlikely that the full meaning of this table is transparent for the reader at the present stage. I apologize for this.

[6]We identify a language with the set of its sentences.

<table>
<tr><th colspan="5">Proving Gärdenfors's theorem</th></tr>
<tr><td colspan="5">Let K be "totally ignorant" about A and B.
Proof idea:
$\boxed{\neg(\mathrm{A\&B}) \Box\!\!\rightarrow \mathrm{A} \in \mathrm{K^o}_{\mathrm{A}} \subseteq \mathrm{K}^{??}_{??} \supseteq \mathrm{K^o}_{\mathrm{B}} \ni \neg(\mathrm{A\&B}) \Box\!\!\rightarrow \mathrm{B}}$
If so, $(\mathrm{K}^{??}_{??})*_{\neg(\mathrm{A\&B})}$ would be inconsistent.
Ideas to get this to work:</td></tr>
<tr><td></td><td colspan="2">theory addition</td><td colspan="2">base addition</td></tr>
<tr><td></td><td colspan="2">$\mathrm{K} \mapsto \mathrm{K^o}_{\mathrm{A}} = \mathrm{Cn}(\mathrm{K} \cup \{\mathrm{A}\})$</td><td colspan="2">$\begin{array}{ccc}\Gamma & \mapsto & \Gamma\cup\{\mathrm{A}\}\\ \mathrm{Cn}\downarrow & & \downarrow\mathrm{Cn}\\ \mathrm{K} & \mapsto & \mathrm{K^o}_{\mathrm{A}}\end{array}$</td></tr>
<tr><td>$\mathbf{K}^{??}_{??} = \ldots$</td><td>$\mathrm{L}_1$</td><td>$\mathrm{L}_2$</td><td>$\mathrm{L}_1$</td><td>$\mathrm{L}_2$</td></tr>
<tr><td rowspan="2">$\mathrm{K^o}_{\{\mathrm{A,B}\}} \supseteq \mathrm{K^o}_{\mathrm{A}}, \mathrm{K^o}_{\mathrm{B}}$</td><td>no</td><td>no</td><td>no</td><td>no</td></tr>
<tr><td colspan="4">since Cn is nonmonotonic</td></tr>
<tr><td rowspan="2">$\mathrm{K^o}_{\mathrm{A\&B}} \supseteq \mathrm{K^o}_{\mathrm{A}}, \mathrm{K^o}_{\mathrm{B}}$</td><td>no</td><td>no</td><td>no</td><td>no</td></tr>
<tr><td colspan="4">since Cn is not classically monotonic</td></tr>
<tr><td>$(\mathrm{K^o}_{\mathrm{A}})^{\mathrm{o}}{}_{\mathrm{B}} = (\mathrm{K^o}_{\mathrm{B}})^{\mathrm{o}}{}_{\mathrm{A}}$</td><td>no</td><td>yes
inconsistent</td><td>yes</td><td>yes</td></tr>
<tr><td>$(\mathrm{K^o}_{\mathrm{A}})^{\mathrm{o}}{}_{\mathrm{B}} \supseteq \mathrm{K^o}_{\mathrm{A}}$</td><td>yes</td><td>yes</td><td>no</td><td>no</td></tr>
<tr><td>Proof of theorem</td><td>fails</td><td>succeeds</td><td>fails</td><td>fails</td></tr>
</table>

to the application of the present logic Cn in belief revision. It is due to the fact that we can model belief revisions and keep the Ramsey test for conditionals without falling prey to the Gärdenfors incompatibility theorem that the present logic is called a logic *for belief revision*. We shall mainly be concerned with belief *additions*, and it will turn out that the method of belief revision advocated violates the preservation principle. We treat additions and revisions not only by "objective" sentences from L_0, but also by conditionals and compounds of conditionals.

In the present paper we confine ourselves to L_1. In order to develop a logic which suits our purposes we do not explicate the relation

$$\Gamma \models \mathrm{A} \ , \ \text{ or equivalently, } \ \mathrm{A} \in \mathrm{Cn}(\Gamma)$$

in the usual way as meaning that *every E-relation which satisfies* Γ *also satisfies A.* This would give us too few consequences of Γ. We restrict the class of E-relations that are suitable for Γ and adopt the following criterion of the preferential-models-approach: *every E-relation that "minimally", or "preferentially", satisfies* Γ *also satisfies A.* In so doing we make Cn nonmonotonic.

The main task then will be to find the right notion of minimality. Three candidates will be considered. The first one will turn out to be insufficient, the second one is quite satisfactory. But we will choose a third one which gives us a unique minimal (in fact a smallest) E-relation for every consistent finite and what is more, for every "well-founded" premise set Γ. In the last section of this paper we examine the inference patterns validated by the resulting conditional logic Cn. In particular we show that the so-called "counterfactual fallacies" (see Lewis 1973, Section 1.8) are *defeasibly valid*, or *valid by default.*

2 Belief revision systems and epistemic entrenchment

Gärdenfors (1988, p. 148) defined a *belief revision system* as a pair $\langle\mathcal{K},*\rangle$ where $\mathcal{K}$ is a set of *belief sets*, i.e., a set of sets of sentences that are closed under the consequence relation Cn_0 of classical propositional logic, and where * is a *belief revision function* taking any belief set K from $\mathcal{K}$ and any L_0-sentence A to the new belief set $*(\mathrm{K},\mathrm{A})\in\mathcal{K}$, or simply $\mathrm{K}*_{\mathrm{A}}$, which is to be interpreted as the minimal revision of K needed to accept A. Moreover, it is required that a belief revision system is rational in the sense that it satisfies a set of rationality postulates originally specified by Gärdenfors in 1982 (see Gärdenfors 1988, Section 3.3). Equivalently, we can say that a belief revision system is a set $\{\langle \mathrm{K},*_{\mathrm{K}}\rangle : \mathrm{K}\in\mathcal{K}\}$, where $*_{\mathrm{K}}$, the revision function associated with K, is obtained by putting $*_{\mathrm{K}}(\mathrm{A})=*(\mathrm{K},\mathrm{A})$ for each $\mathrm{K}\in\mathcal{K}$.

Now let $\mathrm{K}\in\mathcal{K}$ be fixed. Gärdenfors showed that it is, in a very strict sense of the term, the same thing to have a belief revision function $*_{\mathrm{K}}$ for K as it is to have a *belief contraction function* $-_{\mathrm{K}}$ for K satisfying another set of rationality postulates. The relevant connections are furnished by the so-called Levi identity

$$*_{\mathrm{K}}=\mathrm{R}(-_{\mathrm{K}}) \text{ is defined by } \mathrm{K}*_{\mathrm{A}}=\mathrm{Cn}_0(\mathrm{K}^-{}_{\neg\mathrm{A}}\cup\{\mathrm{A}\})$$

and the so-called Harper identity

$$-_K = C(*_K) \text{ is defined by } K^-{}_A = K \cap K*_{\neg A}$$

(see Gärdenfors 1988, Section 3.6). More recently, Gärdenfors and Makinson (1988) showed that it is the same thing to have a contraction function $-_K$ satisfying the relevant set of rationality postulates as it is to have a *relation of epistemic entrenchment*, or shortly an *E-relation*, with respect to K. Now, what are E-relations? An E-relation with respect to K, denoted by $\leq_K$, is a relation holding between L_0-sentences. For $A,B \in L_0$, $A \leq_K B$ is supposed to mean that *B is at least as firmly entrenched in K as A* or, better, *Withdrawing A from K is not harder than withdrawing B.* This can be made quite precise by an idea again due to Gärdenfors. Suppose you are pressed to give up either A or B (where $\nvdash A\&B$), which appears to be the same as to give up A&B. Now you decide to give up A *just in case* B is at least as firmly entrenched in K as A. Since by supposition you have to retract either A or B, this explication clearly entails that $A \leq_K B$ or $B \leq_K A$.

E-relations $\leq_K$ are to satisfy the following conditions (we drop the subscript '$_K$' when there is no danger of confusion):

(E1) If $A \leq B$ and $B \leq C$ then $A \leq C$ (Transitivity)
(E2) If $\emptyset \neq \Gamma \vdash A$ then $B \leq A$ for some $B \in \Gamma$ (Entailment)
(E3) If $B \leq A$ for every B then $\vdash A$ (Maximality)
(E4) If $K \neq L_0$ then $A \leq B$ for every B iff $A \notin K$ (Minimality)

Here and throughout this paper, $\Gamma \vdash A$ is short for $A \in Cn_0(\Gamma)$, $A \vdash B$ is short for $\{A\} \vdash B$ and $\vdash A$ is short for $\emptyset \vdash A$. Condition (E4) expresses the fact that the relation $\leq_K$ of epistemic entrenchment is interesting only within the set K. Outside K, all sentences have equal — viz., minimal — epistemic entrenchment. Condition (E2) replaces Gärdenfors's conditions

(E2a) If $A \vdash B$ then $A \leq B$ (Dominance)
(E2b) $A \leq A\&B$ or $B \leq A\&B$ (Conjunctiveness).

(compare Gärdenfors 1988, Section 4.6, and Gärdenfors and Makinson 1988). It is easily verified that in the presence of (E1) and when applied to belief sets, (E2) is equivalent to the conjunction of (E2a) and (E2b). Apart from reducing the number of postulates, (E2) has two more advantages. First, it has a very clear motivation. For suppose that $\Gamma \vdash A$ and $B \nleq A$ for all $B \in \Gamma$. The latter means, roughly, that it is easier to give up A than give up any B in Γ, which is to say that we may keep all of Γ when removing A. But as A is derivable from Γ by classical propositional logic, we cannot really, or *rationally*, remove A while keeping Γ. In this sense (E2) may be called a rationality criterion. But secondly, note that (E2) makes sense even when K is not closed under Cn_0. Consider for example the set K={A,B,C,A&B&C}. While (E2b) does not apply here, (E2) says that A&B&C is at least as firmly entrenched in K as A or B or C. This is, I believe, in accordance with our intuitions about the rational removal of sentences.

From $\leq$, we define the strict relation $<$ and the equivalence relation $\doteq$ in the usual way: $A<B$ iff $A\leq B$ and $B\not\leq A$, and $A\doteq B$ iff $A\leq B$ and $B\leq A$. Notice that the connectivity condition $A\leq B$ or $B\leq A$ follows from (E1) and (E2). Thus $A<B$ is equivalent to $B\not\leq A$. Other well-known properties of E-relations are the substitutivity of Cn_0-equivalents and the useful

$$A\leq B \quad \text{iff} \quad A\leq A\&B \quad \text{iff} \quad A\doteq A\&B\ .$$

We say that an E-relation $\leq$ is *finite* iff $\doteq$ partitions L_0 into finitely many equivalence classes, and we say that $\leq$ is a *well-ordering E-relation* iff every non-empty set of L_0-sentences has a smallest element under $\leq$. Of course, the well-ordering E-relations include the finite ones. Well-ordering E-relations will play a key role in later sections of this paper. The epistemological drawback of E-relations which are not well-ordering is evident in the case of *multiple* contractions and revisions. When one is forced to give up at least one of the sentences in some set Γ which possesses no smallest element, it is very difficult to see what should be done. No decision to give up one or more certain sentences can be the best decision. People having coarser but well-ordering E-relations are better off.

We have to say how contraction functions $-_K$ are constructed with the help of epistemic entrenchment relations $\leq_K$. In Gärdenfors and Makinson (1988) it is shown that the definitions

$$-_K=C(\leq_K) \text{ is given by } K^-{}_A = \begin{cases} K\cap\{B\colon A<_K A\vee B\} & \text{if } \nvdash A, \\ K & \text{otherwise.} \end{cases}$$

and

$$\leq_K=E(-_K) \text{ is given by } A\leq_K B \text{ iff } A\notin K^-{}_{A\&B} \text{ or } \vdash A\&B.$$

just do the right thing and fit together perfectly. As we will be concerned with revisions only, we take down the direct link between revisions and relations of epistemic entrenchment.

Observation 1 *Let* $R(\leq_K)=_{df}R(C(\leq_K))$ *and* $E(*_K)=_{\mathrm{df}}E(C(*_K))$. *Then*

(i) If $*_K=R(\leq_K)$ *then* $K*_A = \begin{cases} \{B{:}\, \neg A <_K \neg A \vee B\} & \text{if } \nvdash \neg A, \\ L_0 & \text{otherwise.} \end{cases}$

(ii) If $\leq_K=E(*_K)$ *then* $A\leq_K B$ *iff* $A\notin K*_{\neg A\vee\neg B}$ *or* $\vdash A\&B$.

(iii) $R(\leq_K)$ *satisfies the Gärdenfors postulates for revisions if* $\leq_K$ *is an E-relation with respect to K, and* $E(*_K)$ *is an E-relation with respect to K if* $*_K$ *satisfies the Gärdenfors postulates for revisions. Finally,* $R(E(*_K))=*_K$ *and* $E(R(\leq_K))=\leq_K$.

Proofs of the Observations are collected in an appendix at the end of the paper.

It is of crucial importance for the success of this paper that the reader accepts the notion of epistemic entrenchment as a useful and well-considered tool of analysis. First, he or she is recommended to consult the seminal discussions in Gärdenfors (1988, Chapter 4) and Gärdenfors and Makinson (1988). Secondly, it is shown in Rott (1989b) and (1989c) that contractions constructed from relations of epistemic entrenchment are equivalent in a very strict sense to both partial meet contractions (see Alchourrón, Gärdenfors and Makinson 1985) and safe contractions (see Alchourrón and Makinson 1985, 1986). And thirdly, Lindström and Rabinowicz (1990) develop an interesting liberalized notion of epistemic entrenchment with incomparabilies. We take it for granted that contractions and revisions using epistemic entrenchment have a proper standing by now.

So far we have seen that a Gärdenfors belief revision system can be represented by a set $\{\langle \mathrm{K},\leq_{\mathrm{K}}\rangle{:}\ \mathrm{K}\in\mathcal{K}\}$, where $\mathcal{K}$ is a set of belief sets and $\leq_{\mathrm{K}}$ is an E-relation with respect to K, for each $\mathrm{K}\in\mathcal{K}$. To reach our final definition of a belief revision system, we make two more adjustments. In the first step, we note that we can recover every consistent belief set K from $\leq_{\mathrm{K}}$ through

$$\mathrm{K} = \mathrm{K}_0(\leq_{\mathrm{K}}) =_{\mathrm{df}} \{\mathrm{A}{:}\ \bot<_{\mathrm{K}}\mathrm{A}\}\ (\neq\emptyset).$$

That this is true is clear from (E4). So a belief revision system can be represented as $\{\leq_{\mathrm{K}}{:}\ \mathrm{K}\in\mathcal{K}\}$. $\mathcal{K}$ can be treated as an arbitrary index set as long as we remember that for $\mathrm{K}\neq\mathrm{K}'$ we have $\mathrm{K}_0(\leq_{\mathrm{K}})\neq\mathrm{K}_0(\leq'_K)$. It is hard, however, to think of a motivation for this restriction. An E-relation mirrors a person's "objective" beliefs (i.e., beliefs expressible in L_0) as well as his dispositions to change his objective beliefs in response to incoming objective information (recall the definition of $\mathrm{R}(\leq_{\mathrm{K}})$). Two persons then with different relations of epistemic entrenchment are in different epistemic states, even if they agree on the objective beliefs they currently hold. So I suggest as my second step to give up this restriction. Taking E-relations as primitive and belief sets as derived by the equation just mentioned, we can do without belief sets at all. Furthermore, we can drop (E4) from the set of requirements for E-relations. E-relations are no longer E-relations *with respect to some belief set K*, but belief sets are belief sets *obtained from some E-relation* $\leq$. We do not need the index set $\mathcal{K}$ any more. My official definition of a belief revision system reads thus:

Definition 1 *A* belief revision system *is any set* $\mathcal{E}$ *of* E-relations*, i.e., binary relations over* L_0 *satisfying (E1) through (E3). We say that a belief revision system* $\mathcal{E}$ *is* Gärdenforsian *if and only if for every* $\leq$ *and* $\leq'$ *in* $\mathcal{E}$*, if* $K_0(\leq) = K_0(\leq')$ *then* $\leq = \leq'$.[7]

3 Bases for relations of epistemic entrenchment

In the course of this paper we shall often want to discuss concrete examples of E-relations. As E-relations are infinite subsets of $L_0 \times L_0$, this is not a completely trivial matter. What we need is a finite representation of some interesting E-relations which enables us to retrieve the full E-relations in a canonical and easily understandable way. We shall introduce the appropriate means in this section.

Definition 2 *A* base for an E-relation*, or simply, an* E-base*, is a pair* $\langle \mathcal{B}, \preceq \rangle$ *where* $\mathcal{B}$ *is a set of* L_0*-sentences and* $\preceq$ *is a non-strict weak ordering of, i.e., a reflexive, transitive and connected relation over,* $\mathcal{B}$.

[7] We might also call such belief revision systems *functional*. For they specify a unique revision $K{*}_A$ for every $A \in L_0$ and every belief set K such that $\mathcal{E}_K =_{df} \{\leq \in \mathcal{E} : K_0(\leq) = K\}$ is not empty. In general belief revision systems, there are several candidate revisions, one for each $\leq \in \mathcal{E}_K$. This perspective invites interesting comparisons with the work of Lindström and Rabinowicz (1990). For non-empty $\mathcal{E}_K$, for instance, we find that $\bigcap \mathcal{E}_K$ is no E-relation in our sense, but an epistemic entrenchment ordering in the sense of Lindström and Rabinowicz's Definition 3.1. Also see their representation Theorem 3.14. Notice, however, that the "skeptical" intersection of all candidate revisions $\bigcap \{\{B : \neg A < \neg A \vee B\} : \leq \in \mathcal{E}_K\}$ is representable as $\{B : \neg A \vee B \not\leq^* \neg A\}$ where $\leq^* = \bigcup \mathcal{E}_K$. $\leq^*$ is yet another kind of relation (cf. Observation 7 below).

Note that $\mathcal{B}$ need not be consistent and that $\preceq$ need not be antisymmetrical. Given an E-base $\langle\mathcal{B},\preceq\rangle$, a *$\mathcal{B}$-cut* is any subset S of $\mathcal{B}$ such that if A$\in$S and A$\preceq$B then B$\in$S. Since $\preceq$ is connected, $\mathcal{B}$-cuts are nested.

Definition 3 *Let $\langle\mathcal{B},\preceq\rangle$ be an E-base. Then* the E-relation $\leq$=E($\preceq$) generated by $\langle\mathcal{B},\preceq\rangle$ *is given by*

$$A \leq B \quad \textit{iff} \quad \textit{for all } \mathcal{B}\textit{-cuts } S, \textit{ if } A \in Cn_0(S) \textit{ then } B \in Cn_0(S),$$

for all L_0-sentences A and B.

We have to verify that this definition really does what we want.

Observation 2 *Let $\langle\mathcal{B},\preceq\rangle$ be an E-base. Then E($\preceq$) is an E-relation.*

An E-base $\langle\mathcal{B},\preceq\rangle$ can rightly be called a base *for the generated E-relation E($\preceq$)* only if the relationships as specified by $\preceq$ are preserved in E($\preceq$). That is, with $\leq$=E($\preceq$), if for every A and B in $\mathcal{B}$, A$\leq$B if and only if A$\preceq$B, or more succinctly, if $\leq\cap\mathcal{B}\times\mathcal{B} = \preceq$. We would like to know under what circumstances an E-base is a base for its generated E-relation. The following observation demonstrates the usefulness of the Entailment condition.

Observation 3 *An E-base $\langle\mathcal{B},\preceq\rangle$ is a base for E($\preceq$) if and only if $\preceq$ satisfies (E2) over $\mathcal{B}$.*

Notice that if $\preceq$ satisfies (E2) over $\mathcal{B}$ there are in general many E-relations besides E($\preceq$) which preserve the relationships as specified by $\preceq$. These relationships between the sentences in $\mathcal{B}$ might be viewed as providing *partial information* about some underlying full relation of epistemic entrenchment. An E-base $\langle\mathcal{B},\preceq\rangle$, however, is intended to be a means for discussing the unique E-relation E($\preceq$) generated by it.

In the following, we shall use, without any further indication, only E-bases satisfying (E2).

An E-base $\langle\mathcal{B},\preceq\rangle$ is called *finite* if $\mathcal{B}$ is finite. In this case, the relation $\simeq=\preceq\cap\preceq^{-1}$ obviously partitions $\mathcal{B}$ into finitely many equivalence classes. Let the number of equivalence classes be n. We denote the equivalence classes by $\mathcal{B}_i$. The indices are chosen so as to ensure that i$\leq$j iff A$\preceq$B for every A$\in\mathcal{B}_i$ and B$\in\mathcal{B}_j$. We employ the following convenient string notation for $\preceq$:

$$\underbrace{\perp \simeq A_{01} \simeq \ldots \simeq A_{0n_0}}_{\mathcal{B}_0} \prec \underbrace{A_{11} \simeq \ldots \simeq A_{1n_1}}_{\mathcal{B}_1} \prec \ldots$$

$$\ldots \prec \underbrace{A_{m1} \simeq \ldots \simeq A_{mn_m}}_{\mathcal{B}_m} \prec \underbrace{\top}_{\mathcal{B}_{m+1}}$$

where $\prec = \preceq - \simeq$, $m \geq 0$, $n_0 \geq 0$ and $n_i \geq 1$ for $i=1,\ldots,m$. It is understood that $(\mathcal{B} - \mathrm{Cn}_0(\emptyset)) \cup \{\perp, \top\} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_m \cup \mathcal{B}_{m+1}$. If $\mathcal{B} \cap \mathrm{Cn}_0(\emptyset)$ is empty (this will be the case in the intended applications), then $\mathcal{B}_0 = \{\perp\}$ and m=n if $\mathcal{B}$ is consistent, but m=n–1 if $\mathcal{B}$ is inconsistent. If $\mathcal{B} \cap \mathrm{Cn}_0(\emptyset)$ is non-empty, then $\mathcal{B}_0 = \{\perp\}$ and m=n–1 if $\mathcal{B}$ is consistent, but m=n–2 if $\mathcal{B}$ is inconsistent. It is easy to check that the equivalence classes with respect to $\doteq = \mathrm{E}(\preceq) \cap (\mathrm{E}(\preceq))^{-1}$ are given by $\mathrm{Cn}_0(\mathcal{B}_i \cup \mathcal{B}_{i+1} \cup \ldots \cup \mathcal{B}_m) - \mathrm{Cn}_0(\mathcal{B}_{i+1} \cup \mathcal{B}_{i+2} \cup \ldots \cup \mathcal{B}_m)$ for $i=0,\ldots,m$, and $\mathrm{Cn}_0(\emptyset)$.

4 Epistemic entrenchment semantics for conditionals

4.1 Monotonic semantics

Having a precise notion of a belief revision system at his disposal, Gärdenfors was able to develop a formal epistemic semantics for conditionals with the help of the following version of the Ramsey test (R):

(R′) Let $\langle \mathcal{K}, * \rangle$ be a belief revision system in the sense of Gärdenfors. Then, for every $K \in \mathcal{K}$ and every $A,B \in L_0$, $A \Box\!\!\rightarrow B \in K$ iff $B \in *(K,A)$.

By Observation 1, this is equivalent to

$$A \Box\!\!\rightarrow B \in K \quad \text{iff} \quad \neg A <_K \neg A \vee B \ \text{ or } \vdash \neg A.$$

With Definition 1 we modified the concept of a belief revision system by considering E-relations as primitive and allowing one and the same belief set to be associated with several E-relations. Therefore, we will not speak of the inclusion of a conditional in a belief set but of the satisfaction of a conditional by an E-relation.[8]

Definition 4 *An E-relation* $\leq$ satisfies a conditional $A \Box\!\!\rightarrow B$ *iff* $\neg A < \neg A \vee B$ *or* $\vdash \neg A$.

[8]The following definition is formally more similar to Lewis's (1973) evaluation of conditionals than appears at first sight. See Grove (1988) and Gärdenfors (1988, Section 4.8).

The principal condition $\neg A<\neg A\vee B$, i.e., $\neg A<A\to B$, can be motivated as follows. When an E-relation $\leq$ says that the material conditional $A\to B$ is more firmly entrenched than $\neg A$, this can be taken to mean that the material conditional is accepted not just because the negation of the antecedent is accepted. And more, if a person in epistemic state $\leq$ should come to learn that A is in fact true, this would not destroy his or her belief in $A\to B$. Put as a slogan, a natural language conditional is the corresponding material conditional believed more firmly than the negation of its antecedent. Note that conditionals express strict $<$-relationships, not non-strict $\leq$-relationships. In view of Observation 1, $A<B$ is expressible by means of the L_1-sentence $\mathsf{If}\ \neg A\vee\neg B\ \mathsf{then}\ B$. But only in L_2 will we dispose of a linguistic expression for $A\leq B$.

An E-relation $\leq$ is said to *satisfy an L_0-sentence* A iff A is in $K_0(\leq)$, i.e., iff $\bot<A$.[9] By (E1)–(E3), $K_0(\leq)$ is consistent and closed under Cn_0, for every E-relation $\leq$. E-relations are *non-classical models*, since, e.g., it is not the case that $\leq$ satisfies $\neg A$ iff $\leq$ does not satisfy A. Nor can E-relations be regarded as the models of a three-valued "truth-functional" logic with the values 'accepted', 'rejected' and 'undecided', because it is impossible to determine the value of $A\vee B$ from the values of A and B if the latter are both 'undecided'. It is either 'undecided' or 'accepted'.[10]

If an E-relation $\leq$ satisfies an L_1-sentence A we write $\leq\models A$, and we set

$$K(\leq) =_{\mathrm{df}} \{A\in L_1 : \leq\models A\}.$$

Sometimes we say that $K(\leq)$ is the *belief set* or the *theory* associated with the E-relation $\leq$. Obviously, $K(\leq)\cap L_0=K_0(\leq)$. An E-relation *satisfies a set* Γ *of L_1-sentences* if it satisfies every element of Γ, i.e., if $\Gamma\subseteq K(\leq)$. Sometimes, when Γ is a given premise set, we say that $\leq$ is an *E-relation for* Γ iff $\leq$ satisfies Γ. More semantic concepts are readily defined along the standard lines:

Definition 5 *An L_1-sentence A is called* satisfiable *or* consistent *if it is sat-*

[9] As regards satisfaction, an "objective" L_0-sentence A is equivalent to the conditional $\top\Box\!\!\to A$. But they differ in syntactic behaviour. In L_1, we have for instance $A\Box\!\!\to A$ but not $(\top\Box\!\!\to A)\Box\!\!\to(\top\Box\!\!\to A)$. Moreover, in Part 2, we shall argue that $\neg A$ differs from $\neg(\top\Box\!\!\to A)$ in meaning.

[10] Like belief sets, E-relations seem to obey the logic of supervaluations instead. Cf. Martin (1984).

isfied by some E-relation,[11] *and A is called* (monotonically) valid*, in symbols* $\models_1$A*, if it is satisfied by every E-relation.* Γ *is said to be* consistent *if there is an E-relation for* Γ*. An* L_1*-sentence A is* (monotonically) entailed *by a set* Γ *of* L_1*-sentences, in symbols* $\Gamma\models_1$A *or equivalently* A$\in$Cn$_1(\Gamma)$*, if every E-relation satisfying* Γ *also satisfies A.*

Example 1 Now, at last, we are able to deal with the introductory beach walk and the generalizations we drew from it. Remember that we have set out to find a way to get (no — not a hamburger, but) the paradigmatic inference[12]

$$\{\mathrm{A}\vee\mathrm{B}\} \models \neg\mathrm{A}\Box\!\!\rightarrow\mathrm{B} \tag{3}$$

and yet block the inference

$$\{\mathrm{A}\vee\mathrm{B},\mathrm{A},\neg\mathrm{B}\} \models \neg\mathrm{A}\Box\!\!\rightarrow\mathrm{B} \tag{4}$$

Let us see if our logic Cn$_1$ is appropriate. The inference (4) is indeed blocked. Consider the E-relation $\leq$ generated by the E-base

$$\bot \prec \mathrm{A}\vee\mathrm{B} \simeq \mathrm{A} \simeq \neg\mathrm{B} \prec \top$$

Obviously, $\leq$ satisfies all the premises of (4), and it does so in an intuitively plausible way. In order to satisfy the conclusion of (4), $\leq$ would have to be such that A$<$A$\vee$B holds. There would have to be a $\mathcal{B}$-cut S such that A$\vee$B but not A is in Cn$_0$(S). But there is none.

Next consider (3). The most natural E-relation for the single premise A$\vee$B, viz., that generated by the E-base

$$\bot \prec_1 \mathrm{A}\vee\mathrm{B} \prec_1 \top\ ,$$

behaves well. It indeed yields A$<_1$A$\vee$B, since A$\vee$B but not A is a Cn$_0$-consequence of $\mathrm{S}_{\mathrm{A}\vee\mathrm{B}}=_{\mathrm{df}}\{\mathrm{C}\in\mathcal{B}\colon \mathrm{A}\vee\mathrm{B}\preceq_1\mathrm{C}\}=\{\mathrm{A}\vee\mathrm{B}\}$. But of course there are more E-relations satisfying A$\vee$B, for example the one generated by the E-base

$$\bot \prec_2 \mathrm{A} \prec_2 \top\ .$$

[11] Recall that E-relations themselves, or rather their L$_0$-images K$_0(\leq)$, are always consistent.

[12] In these and all similar considerations to follow, it is understood that A and B are contingent L$_0$-sentences which are independent with respect to Cn$_0$.

And it gets clear immediately that the E-relation $E(\preceq_2)$ does not satisfy $\neg A \Box\!\!\rightarrow B$. So we cannot validate (3), if $\models_1$ is substituted for $\models$. (End of example)

The objection to this last line of reasoning is that there is nothing which could justify the E-base $\langle\{A\},\preceq_2\rangle$ if all we know is $A \vee B$. In every conceivable sense, the E-base $\langle\{A \vee B\},\preceq_1\rangle$ is much more natural for the singleton premise set $\{A \vee B\}$ than $\langle\{A\},\preceq_2\rangle$. Among the E-relations satisfying some premise set Γ, it appears, there are E-relations that are appropriate for Γ and E-relations that are inappropriate for Γ. There exists, one may suppose, a *preference ordering* among the E-relations satisfying Γ. And only the best E-relations matter. If all of the best ones satisfy the conclusion of the inference, then the inference may be called "valid". Section 4.2 will reveal that we have just argued for employing the techniques of a quite well-known kind of non-monotonic logic.

Before turning to this abstract topic, let us remain at the paradigmatic infernce patterns (3) and (4) for a moment. The problem was found to lie in the validation of (3). What is it that makes $E(\preceq_2)$ so much worse for the single premise $A \vee B$ than $E(\preceq_1)$? It is safe to assume that in this particular case where $A \vee B$ is supposed to be *all one knows*, the relationship $\bot < A$ is not warranted. But what is the general mistake? Three suspicions come to one's mind.

- $E(\preceq_2)$ satisfies *too many sentences*. In order not to invoke "beliefs" that are not justified by the premise set, we should try to minimize the set of sentences satisfied by an appropriate E-relation for the premise set. Just as in usual monotonic logics the deductive closure of a set Γ is the minimal theory including Γ, we should opt for minimal theories (associated with some E-relation) including Γ in the present case.

- $E(\preceq_2)$ satisfies *too many L_0-sentences.* The motivation for this idea is the same as for the last one. If it should turn out insufficient to minimize the number of L_1-sentences (L_1 is the language under consideration), it seems plausible to attribute a preferred status to the "objective" L_0-sentences.

- $E(\preceq_2)$ assigns to some L_0-sentence, for example to A, a *gratuitously high "rank" of epistemic entrenchment* which is not justified by the single premise $A \vee B$. It seems prudent not to attribute a greater degree of irremovability to any objective sentence than is explicitly warranted by the premise set. A believer should be prepared to give up his or her beliefs by minimizing epistemic entrenchments.

I think that all three of these suggestions have a sound basis. In Sections 5–7, we shall examine the consequences of taking them into accout within the framework of nonmonotonic reasoning we are now going to introduce.

4.2 Nonmonotonic semantics

We said that when inquiring whether A follows from a given premise set Γ we only want to consider the *preferred* E-relations satisfying Γ. In the three informal objections against the inadequate E-base which invalidates (3) we found that we wanted to *minimize* certain parameters of E-relations satisfying Γ. These formulations will ring a bell in the ears of those acquainted with the work that has been done in the field of nonmonotonic reasoning. In fact, we can draw on the *minimal models approach* or *preferential models approach* which was developed in its general form by Shoham (1987, 1988). Makinson (1989) generalized it to cases in which the models considered are allowed to be, like E-relations, non-classical. We now adapt some of their central definitions to our purposes of providing an epistemic semantics for conditionals.

Definition 6 *Let $\sqsubset$ be a strict partial ordering of (i.e., an asymmetric and transitive relation over) the class of all E-relations. Then an E-relation $\leq$ is called* minimal *(or* preferred*) iff there is no E-relation $\leq'$ such that $\leq'\sqsubset\leq$.*[13] *Let Γ be a set of L_1-sentences. An E-relation $\leq$ is called* minimal for Γ *if $\leq\models\Gamma$ and there is no E-relation $\leq'$ such that $\leq'\models\Gamma$ and $\leq'\sqsubset\leq$. In this case we say that $\leq$* minimally *(or* preferentially*)* satisfies Γ *(with respect to $\sqsubset$) and write $\leq\models_\sqsubset\Gamma$. We say that A is* minimally valid*, in symbols $\models_\sqsubset \mathrm{A}$, if every minimal E-relation satisfies A. We say that* Γ minimally entails A*, in symbols $\Gamma\models_\sqsubset\mathrm{A}$, if every minimal E-relation for Γ satisfies A. We also write* $\mathrm{Cn}_\sqsubset(\Gamma)$ *for $\{A\in L_1 : \Gamma\models_\sqsubset A\}$.*

Be aware that if $\sqsubset\subseteq\sqsubset'$ then there are at least as many $\sqsubset$-minimal E-relations as $\sqsubset'$-minimal ones, and hence $\mathrm{Cn}_\sqsubset(\Gamma)\subseteq\mathrm{Cn}_{\sqsubset'}(\Gamma)$. The intuitive idea behind

[13]This is not a very interesting definition. With respect to the three orderings for E-relations suggested in the next section, there is only one smallest E-relation, viz., that generated by the E-base $\bot\prec\top$.

preferential entailment in our case is that only a minimal E-relation for Γ is an epistemic state which is warranted if all the items one explicitly knows are given by Γ. And only warranted belief states should be called upon when determining the consequences of a premise set. The task before us now is to explicate what features can make an E-relation count as "minimal" or "preferred".

From the non-monotonic point of view, it is interesting to enquire the circumstances under which an inference is *robust* (or *persistent* or *stable*) under possible enrichments of the premise set. We say that Γ *robustly entails* A (with respect to some given $\sqsubset$), in symbols $\Gamma \models\!\equiv_{\sqsubset} A$, iff $\Gamma \models_{\sqsubset} A$ and for every superset Σ of Γ, $\Sigma \models_{\sqsubset} A$. It turns out that normally, and in particular in our concrete instantiations of $\sqsubset$ presented below, $\models\!\equiv_{\sqsubset}$ is just identical with the old monotonic consequence relation $\models_1$:

Observation 4 *Let $\sqsubset$ be a strict partial ordering of the class of E-relations, Γ be a set of L_1-sentences and A an L_1-sentence. Then*

(i) If $\Gamma \models_1 A$ then $\Gamma \models\!\equiv_{\sqsubset} A$.

(ii) If every E-relation $\leq$ is $\sqsubset$-minimal for $K(\leq)$, then if $\Gamma \models\!\equiv_{\sqsubset} A$ then $\Gamma \models_1 A$.

It is natural to assume that every E-relation $\leq$ is among the preferred E-relations for the total set K($\leq$) of sentences satisfied by $\leq$. We shall find that this assumption is fulfilled in all three orderings of E-relations to be discussed in the next section.[14]

5 Three orderings for relations of epistemic entrenchment

In this section, we are going to work out the details of the three suggestions that were made in response to the failure of (3) in the monotonic setting of Cn_1. The first one was that an E-relation which is "grounded in" or "induced by" a given premise set Γ should not satisfy more sentences than necessary. That is, an E-relation $\leq$ for Γ is better than another E-relation $\leq'$ for Γ if K($\leq$) is a proper subset of K($\leq'$).

[14]See Definition 7, Observation 6, Observation 13 and its corollary.

Definition 7 *Let $\leq$ and $\leq'$ be E-relations. Then $\leq$ is* at least as K-good *as $\leq'$, in symbols $\leq \sqsubseteq_{\mathrm{K}} \leq'$, if and only if $K(\leq) \subseteq K(\leq')$. $\leq$ is* K-preferred *over $\leq'$, in symbols $\leq \sqsubset_{\mathrm{K}} \leq'$, if and only if $\leq \sqsubseteq_{\mathrm{K}} \leq'$ and not $\leq' \sqsubseteq_{\mathrm{K}} \leq$.*[15]

As we are taking E-relations, rather than belief sets, as primary representations of epistemic states, it is desirable to replace this metamathematical definition referring to sets of sentences and satisfaction by a purely mathematical condition.

Observation 5 *Let $\leq$ and $\leq'$ be E-relations. Then the following conditions are equivalent:*

(i) $\leq \sqsubseteq_{\mathrm{K}} \leq'$;
(ii) $\leq' \subseteq \leq$;
(iii) $< \subseteq <'$;
(iv) $\doteq' \subseteq \doteq$.

An obvious corollary is

Corollary *Let $\leq$ and $\leq'$ be E-relations. Then the following conditions are equivalent:*

(i) $\leq \sqsubset_{\mathrm{K}} \leq'$;
(ii) $\leq' \subset \leq$;
(iii) $< \subset <'$;
(iv) $\doteq' \subset \doteq$.

Now we have got quite a good picture of what K-preference consists in. An E-relation $\leq$ satisfies less L_1-sentences than another E-relation $\leq'$ if and only if $\doteq' \subset \doteq$. This means that whenever two L_0-sentences A and B are in the same equivalence class with respect to $\doteq'$ then they are in the same equivalence class with respect ot $\doteq$, and besides there are L_0-sentences A and B which are equivalent with respect to $\doteq$ but not with respect to $\doteq'$. $\doteq$ is a coarsening of $\doteq'$. If $\doteq'$ is given by an E-base in string notation, then a K-preferred $\doteq$ is obtained by replacing one or more occurances of $\prec$ in the string by $\simeq$. We can rephrase the idea of K-preference as follows: *Choose as coarse an E-relation (for a given premise set Γ) as possible! Do not impose unnecessary differences in the degrees of epistemic entrenchment!*

[15]The reader be warned that the direction of '$\sqsubseteq$' and '$\sqsubset$' may be the reverse of what he or she has expected. The reason for this is that the preferred E-relations are, in some intuitive as well as formal sense, minimal.

Plausible as all this may be, it is not sufficient. This is borne out dramatically by our paradigm Example 1, where for Γ={A$\vee$B} the E-bases $\bot\prec_1 A\vee B\prec_1\top$ and $\bot\prec_2 A\prec_2\top$ both generate minimal E-relations for Γ with respect to $\sqsubset_K$. For any attempt to extend $\simeq_1$ and $\simeq_2$ will result in the trivial base $\bot\prec\top$ and thus fail to satisfy Γ. Note in particular that E($\prec_1$) and E($\prec_2$) are incomparable with respect to $\sqsubset_K$, since $\neg A\Box\!\!\rightarrow B$ is in K(E($\preceq_1$))–K(E($\preceq_2$)) and A is in K(E($\preceq_2$))–K(E($\preceq_1$)). We do not get $\Gamma\models_{\sqsubset_K}\neg A\Box\!\!\rightarrow B$. Preference with respect to $\sqsubset_K$, therefore, cannot be the key for the validation of (3).

But clearly, E($\preceq_1$) should be preferred to E($\preceq_2$) in Example 1. It seems obvious that the defect of $\preceq_2$ as an E-base for Γ={A$\vee$B} lies in the fact that $\bot\prec_2 A$, i.e., that A is satisfied by E($\preceq_2$). There is no reason for this to be found in Γ. So we turn to the second idea propounded at the end of Section 4.1, namely that E($\preceq_2$) satisfies too many L_0-sentences. In order to further compare the E-relations E($\preceq_1$) and E($\preceq_2$) even though K(E($\preceq_1$)) and K(E($\preceq_2$)) are not related by set inclusion, we adopt the following maxim: *Among the K-minimal E-relations, choose only those that commit us to as few L_0-sentences as possible! Do not adopt unwarranted "objective" beliefs!*

Definition 8 *Let $\leq$ and $\leq'$ be E-relations. Then $\leq$ is* at least as K_0-good *as $\leq'$, in symbols $\leq\sqsubseteq_{K_0}\leq'$, if and only if $\leq\sqsubseteq_K\leq'$, or $\leq$ and $\leq'$ are $\sqsubseteq_K$-incomparable and $K_0(\leq)\subseteq K_0(\leq')$. $\leq$ is* K_0-preferred *over $\leq'$, in symbols $\leq\sqsubset_{K_0}\leq'$, if and only if $\leq\sqsubseteq_{K_0}\leq'$ and not $\leq'\sqsubseteq_{K_0}\leq$.*

The definition of $\sqsubseteq_{K_0}$ is a bit complicated. Fortunately, the strict version $\sqsubset_{K_0}$ which is the one that in fact enters into the nonmonotonic semantical apparatus, is captured by a nice and easy condition.

Observation 6 *Let $\leq$ and $\leq'$ be E-relations. Then*
$\leq\sqsubset_{K_0}\leq'$ iff $K(\leq)\subset K(\leq')$ or $K_0(\leq)\subset K_0(\leq')$.

Note that $\sqsubset_{K_0}$ is transitive because $K(\leq)\subset K(\leq')$ implies $K_0(\leq)\subseteq K_0(\leq')$. Being an extension of $\sqsubset_K$, $\sqsubset_{K_0}$ allows us to compare more E-relations than the former. As a consequence, $Cn_{\sqsubset_{K_0}}(\Gamma)$ is a superset of $Cn_{\sqsubset_K}(\Gamma)$ for every premise set Γ. In most cases, the latter will be a *proper* subset of the former. This is true in Example 1, which K_0-preference gets right. It is evident that $\bot\prec_1 A\vee B\prec_1\top$ is the base of the only minimal E-relation for Γ={A$\vee$B} with respect to $\sqsubset_{K_0}$, so we have $\Gamma\vdash_{\sqsubset_{K_0}}\neg A\Box\!\!\rightarrow B$. We have managed to find a plausible way of validating the desired inference (3).

It is an interesting and important question whether K_0-preference gives, as in Example 1, always a unique E-relation for a (finite) premise set Γ. The answer is no:

Example 2 Let $\Gamma=\{A,B,\neg A \Box\!\!\rightarrow B\}$. Then the E-bases $\bot \prec A \simeq B \prec A\vee B \prec \top$ and $\bot \prec' A \prec' B \simeq' A\vee B \prec' \top$ both generate E-relations $\leq = E(\preceq)$ and $\leq' = E(\preceq')$ for Γ. Both $\leq$ and $\leq'$ are K-minimal for Γ. Any attempt to reduce the number of equivalence classes of $\leq$ and $\leq'$ will result in a violation of either $\bot <^{(\prime)} A$, $\bot <^{(\prime)} B$ or $A <^{(\prime)} A\vee B$. In particular, $\leq$ and $\leq'$ are incomparable with respect to $\sqsubseteq_K$. On the one hand, $\leq$ satisfies but $\leq'$ does not satisfy $\neg B \Box\!\!\rightarrow A$, on the other hand, $\leq'$ satisfies but $\leq$ does not satisfy $\neg A \vee \neg B \Box\!\!\rightarrow B$. Furthermore, $K_0(\leq)=Cn_0(\{A\&B\})=K_0(\leq')$, and obviously any E-relation for Γ must satisfy $Cn_0(\{A\&B\})$. Hence both $\leq$ and $\leq'$ are K_0-minimal for Γ. It is straightforward to check that $\leq$ and $\leq'$ are the only K_0-preferred E-relations for Γ. (End of example)

Intuitively, K_0-preference seems to be a very natural ordering of E-relations. Still there is an objection. We know from Example 2 that in general there is more than one minimal E-relation with respect to $\sqsubset_{K_0}$ for a given premise set Γ. The consequences of Γ, according to $Cn_{\sqsubset_{K_0}}$, are those L_1-sentences which are satisfied by all $\sqsubset_{K_0}$-minimal E-relations for Γ, i.e., $\bigcap\{K(\leq)\colon \leq \text{ is } \sqsubset_{K_0}\text{-minimal for } \Gamma\}$. The question arises as to what *the* epistemic state is, if Γ is all one explicitly knows and Γ admits various $\sqsubset_{K_0}$-minimal candidates. It turns out that in most cases it cannot be an E-relation. To see this, we define $K(\leq)=\{A\in L_1\colon \leq\models A\}$ for an *arbitrary* binary $\leq$ over L_0, to be the set $\{A\in L_0\colon A\not\leq\bot\}\cup\{B\Box\!\!\rightarrow C\in L_1\colon \neg B\vee C\not\leq\neg B\}$.[16]

Observation 7 *Let $\leq_1, \ldots, \leq_n$ be E-relations and $\leq\ =\ \leq_1\cup\ldots\cup\leq_n$. Then*

(i) $K(\leq) = K(\leq_1)\cap\ldots\cap K(\leq_n)$.

(ii) If $\leq^*$ is an E-relation such that $K(\leq^*) = K(\leq_1)\cap\ldots\cap K(\leq_n)$, then $\leq^*=\leq$.

(iii) $\leq$ fails to be an E-relation iff there are sentences A,B,C in L_0 such that $A<_i B\leq_i C$ and $C\leq_j A<_j B$ for some $1\leq i,j\leq n$, and $A<_k B$ for every $1\leq k\leq n$.

From this observation it is clear that if there are multiple $\sqsubset_{K_0}$-minimal E-relations for a premise set Γ, we cannot expect to have a unique E-relation $\leq^*$

[16] I always presuppose that the satisfaction of an L_1-sentence is defined for non-E-relations $\leq$ in the same way as for E-relations. Different, more complicated definitions of satisfaction may make a big difference.

that satisfies exactly all those sentences satisfied by each of them. Given two E-relations Γ_1 and Γ_2, in particular, we will in most cases find L_0-sentences A,B,C such that $A <_1 B \leq_1 C$ and $C \leq_2 A <_2 B$ (or vice versa). For $\leq^* = E(\preceq) \cup E(\preceq')$, we have in Example 2 $A \vee B \leq^* B$ and $B \leq^* A$, but not $A \vee B \leq^* A$, and, for the sake of illustration, in Example 1 we find that for $\leq^* = E(\preceq_1) \cup E(\preceq_2)$ $A \vee B \leq^* A$ and $A \leq^* \bot$, but not $A \vee B \leq^* \bot$. Violation of transitivity seems to be the rule rather than the exception.

We would like to identify $Cn(\Gamma)$ with the set of sentences accepted by an idealized "rational" believer whose only explicit information is given by Γ. We have seen, however, that if we take $Cn_{\sqsubset_{K_0}}$ as Cn, there is in general no E-relation satisfying all and only the sentences in $Cn_{\sqsubset_{K_0}}(\Gamma)$. Hence the believer's beliefs cannot be mirrored by an E-relation. This is an abstract problem as yet, concerning the formal representation of belief states. Why not just give up the doctine that an epistemic state is best represented by a single relation of epistemic entrenchment? In fact, in Part 2 of the present trilogy, we shall have to give up this doctrine anyway when considering disjunctions of conditionals. Moreover, in Part 3, we shall argue that when it comes to belief revision, it is not E-relations but premise sets which should be taken as the primary objects of revision.

Yet we stick to the thesis that an epistemic state should be represented by a single E-relation in this paper. First, it seems reasonable to assume that something like a measure of the firmness of belief is transitive. Secondly, it is easily verified that in terms of the conditionals satisfied, the transitivity condition (E1) is equivalent to

$$\text{If } \leq \models A \vee C \Box\!\!\rightarrow \neg C \text{ and } \leq \not\models B \vee C \Box\!\!\rightarrow \neg C \text{ then } \leq \models A \vee B \Box\!\!\rightarrow \neg B. \qquad \text{(Transitivity by Conditionals)}$$

Substituting D for A, $\neg E$ for B and $D \& \neg E$ for C, we see that Transitivity by Conditionals entails

$$\text{If } \leq \models D \Box\!\!\rightarrow E \text{ and } \leq \not\models \neg E \Box\!\!\rightarrow \neg D \text{ then } \leq \models D \vee \neg E \Box\!\!\rightarrow E. \qquad \text{(Failure of Contraposition[17])}$$

If one wants to retain these conditions *for epistemic states*, then one cannot opt for the transition to unions of E-relations. In Example 2, for instance,

[17] Notice that $\leq \models D \Box\!\!\rightarrow E$ iff $\leq \models D \Box\!\!\rightarrow \neg D \vee E$. — We shall return to the failure of contrapostion for conditionals in Section 7 below and in Part 2 of the trilogy.

we find that $\leq$*=E($\preceq$)∪E($\preceq'$) satisfies ¬A□→B, but neither ¬B□→A nor ¬A∨¬B□→B. I have to admit, though, that the intuitions behind such inference patterns are not very strong.

The main reason for my tentative insistence on the *one E-relation doctrine* derives from the third idea put forward at the end of Section 4.1. We shall presently show that if we do not assign greater ranks of epistemic entrenchment to L_0-sentences than is explicitly required by a given (well-behaved) premise set Γ, then Γ "induces" a *unique* minimal, and in fact a smallest, E-relation for Γ. In sum, then, I do not want to say that K_0-preference is not good, but I put it aside only because I think that there is a more promising alternative.

In a way, this alternative just generalizes on the idea of K_0-preference. Opting for an E-relation which is minimal with respect to $\sqsubset_{K_0}$ means opting for a maximal set of sentences with the lowest epistemic rank possible, viz., the rank of ⊥. But why follow the prudent strategy of accepting things just to the degree they are explicitly warranted only at this lowest level? It seems to me that believers are well-advised if they adopt the distrustful *maxim of universal minimality*: *Do not have more confidence in your items of belief than is assured by your premises! Assign to all sentences the lowest epistemic rank possible!*

In order to make this idea more precise we need the notion of the *rank of epistemic entrenchment* of a sentence A according to an E-relation $\leq$. This notion makes sense for well-ordering E-relations.

Definition 9 *Let $\leq$ be a well-ordering E-relation. Then we define for any ordinal α*

$$\alpha(\leq) = \{A \in L_0 - \overleftarrow{\alpha}(\leq) \colon A \leq B \text{ for all } B \in L_0 - \overleftarrow{\alpha}(\leq)\},$$
$$\text{where } \overleftarrow{0}(\leq) =_{\mathrm{df}} \emptyset \text{ and } \overleftarrow{\alpha}(\leq) =_{\mathrm{df}} \bigcup\{\beta(\leq) \colon \beta < \alpha\} \text{ for } \alpha > 0.$$

Then for every L_0-sentence A, $\mathrm{rank}_{\leq}(A) = \alpha$ *iff* $A \in \alpha(\leq)$.

As we can go on with this construction up to any arbitrary ordinal, $\mathrm{rank}_{\leq}$ is well-defined for well-ordering E-relations even if L_0 is supposed to have nondenumerably many atoms. And by construction, if there is no $A \in L_0$ such that $\mathrm{rank}_{\leq}(A) = \alpha$ for an ordinal α, then there is no $B \in L_0$ such that $\mathrm{rank}_{\leq}(B) = \beta$ for any $\beta > \alpha$. All ranks are "occupied". It is also easy to see that $A \leq B$ if and only if $\mathrm{rank}_{\leq}(A) \leq \mathrm{rank}_{\leq}(B)$. If a finite E-relation is generated by an E-base

$$\underbrace{\bot \simeq A_{01} \simeq \ldots \simeq A_{0n_0}}_{\mathcal{B}_0} \prec \underbrace{A_{11} \simeq \ldots \simeq A_{1n_1}}_{\mathcal{B}_1} \prec \ldots$$

$$\ldots \prec \underbrace{A_{m1} \simeq \ldots \simeq A_{mn_m}}_{\mathcal{B}_m} \prec \underbrace{\top}_{\mathcal{B}_{m+1}}$$

satisfying (E2) over $\mathcal{B}$ then rank$_{\leq}$(A$_{ij}$)=i, as expected. More generally, rank$_{\leq}$(A)=i for any A in Cn$_0$($\mathcal{B}_i \cup \mathcal{B}_{i+1} \cup \ldots \cup \mathcal{B}_m$)–Cn$_0$($\mathcal{B}_{i+1} \cup \mathcal{B}_{i+2} \cup \ldots \cup \mathcal{B}_m$). We can now formulate a precise definition for the new maxim.

Definition 10 *Let $\leq$ and $\leq'$ be E-relations. Then $\leq$ is* at least as E-good *as $\leq'$, in symbols $\leq \sqsubseteq_{\mathrm{E}} \leq'$, if and only if $\leq$ is well-ordering and $\leq'$ is not, or both $\leq$ and $\leq'$ are well-ordering and rank$_{\leq}$(A)$\leq$rank$_{\leq'}$(A) for every L_0-sentence A. $\leq$ is* E-preferred *over $\leq'$, in symbols $\leq \sqsubset_{\mathrm{E}} \leq'$, if and only if $\leq \sqsubseteq_{\mathrm{E}} \leq'$ and not $\leq' \sqsubseteq_{\mathrm{E}} \leq$.*

It is easy to check that $\sqsubseteq_{\mathrm{E}}$ is antisymmetrical. Now our first task is to explore the relationship between E-preference and K-preference and K$_0$-preference.

Observation 8 *Within the class of well-ordering E-relations, $\sqsubseteq_{\mathrm{K}} \subseteq \sqsubseteq_{\mathrm{E}}$.*

Corollary *Within the class of well-ordering E-relations, $\sqsubset_{\mathrm{K}} \subseteq \sqsubset_{\mathrm{E}}$.*

$\sqsubset_{\mathrm{E}}$ is an extension of $\sqsubset_{\mathrm{K}}$ just as $\sqsubset_{\mathrm{K}_0}$ was. The relation between $\sqsubset_{\mathrm{E}}$ and $\sqsubset_{\mathrm{K}_0}$, on the other hand, is more delicate. There are examples of E-relations $\leq$ and $\leq'$, for which $\leq \sqsubset_{\mathrm{K}_0} \leq'$ but not $\leq \sqsubset_{\mathrm{E}} \leq'$, such as those based on

$$\begin{array}{ccccccccc} \bot & \prec & \mathrm{A} & \prec & \mathrm{B} & \prec & \top & \text{and} \\ \bot & \prec' & \mathrm{B} \simeq' \mathrm{C} & \prec' & \mathrm{A} & \prec' & \top & , \end{array}$$

and also examples where the converse holds, such as those based on

$$\begin{array}{ccccccccc} \bot & \prec & \mathrm{A} \simeq \mathrm{B} & \prec & \mathrm{C} & \prec & \top & \text{and} \\ \bot & \prec' & \mathrm{A} & \prec' & \mathrm{B} \simeq' \mathrm{C} & \prec' & \top & . \end{array}$$

The best we can do is state is the following

Observation 9 *Within the class of well-ordering E-relations, $\sqsubseteq_{\mathrm{E}} \subseteq \sqsubseteq_{\mathrm{K}_0}$.*

6 Constructing E-minimal relations of epistemic entrenchment for well-founded premise sets

We decide to base the following considerations on $\sqsubset_{\mathrm{E}}$. In this section we are going to show that with respect to $\sqsubset_{\mathrm{E}}$, every consistent finite set of L_1-premises possesses a unique minimal, and in fact a smallest, E-relation satisfying it. This allows us to keep the one-E-relation doctrine for all practical applications of L_1. We shall also consider the case of an infinite Γ.

An arbitrary set Γ of L_1-sentences can be given the following format. It divides into a set Γ_0 of L_0-premises A_i and a set Γ_1 of conditionals from L_1–L_0 of the form $\mathrm{B}_j \Box\!\!\rightarrow \mathrm{C}_j$. When trying to find an E-relation satisfying Γ, one can regard the premises as providing partial information about the set of E-relations — or preferably, about *the* E-relation — constituting the epistemic state of an individual whose only explicit information consists in Γ. For the sake of simplicity, we cancel all conditionals $\mathrm{B}_i \Box\!\!\rightarrow \mathrm{C}_i$ for which $\neg\mathrm{B}_i \in \mathrm{Cn}_0(\emptyset)$. By Definition 4, these conditionals are satisfied by every E-relation, so they do not matter. Recalling how satisfaction of L_1-sentences by E-relations has been defined, we can now describe the situation with the following figure:

$$
\begin{array}{ccc}
\boxed{\quad\Gamma\quad} & \longmapsto & \boxed{\quad\leq\quad} \\
\mathrm{A}_1 & \longmapsto & \bot < \mathrm{A}_1 \\
\mathrm{A}_2 & \longmapsto & \bot < \mathrm{A}_2 \\
\mathrm{A}_3 & \longmapsto & \bot < \mathrm{A}_3 \\
\vdots & \vdots & \vdots \\
\mathrm{B}_1 \Box\!\!\rightarrow \mathrm{C}_1 & \longmapsto & \neg\mathrm{B}_1 < \mathrm{B}_1 \rightarrow \mathrm{C}_1 \\
\mathrm{B}_2 \Box\!\!\rightarrow \mathrm{C}_2 & \longmapsto & \neg\mathrm{B}_2 < \mathrm{B}_2 \rightarrow \mathrm{C}_2 \\
\mathrm{B}_3 \Box\!\!\rightarrow \mathrm{C}_3 & \longmapsto & \neg\mathrm{B}_3 < \mathrm{B}_3 \rightarrow \mathrm{C}_3 \\
\vdots & \vdots & \vdots
\end{array}
$$

Evidently, an L_0-sentence A_i has the same satisfaction condition as the corresponding L_1-sentence $\top \Box\!\!\rightarrow \mathrm{A}_i$. Another simplifying move consists in identifying objective sentences with their conditional counterparts. We can therefore assume that every premise set Γ in L_1 is a set $\{\mathrm{A}_i \Box\!\!\rightarrow \mathrm{B}_i \colon \mathrm{i} \in \mathrm{I}\}$ of conditionals where I is a possibly infinite index set and $\neg A_i \notin Cn_0(\emptyset)$ for every $i \in I$. It will be helpful to have in mind a seperate picture for the simplified format:

$$
\begin{array}{ccc}
\boxed{\quad\Gamma\quad} & \longmapsto & \boxed{\quad\leq\quad} \\
A_1 \Box\!\!\rightarrow B_1 & \longmapsto & \neg A_1 < A_1 \rightarrow B_1 \\
A_2 \Box\!\!\rightarrow B_2 & \longmapsto & \neg A_2 < A_2 \rightarrow B_2 \\
A_3 \Box\!\!\rightarrow B_3 & \longmapsto & \neg A_3 < A_3 \rightarrow B_3 \\
\vdots & \vdots & \vdots
\end{array}
$$

Now the construction of an E-minimal E-relation for Γ, i.e., of an E-relation which assigns to all sentences the lowest epistemic rank possible, is pretty obvious. In the first step we note that the partial information about admissible E-relations provided by Γ "forces" all material conditionals $A_i \rightarrow B_i$ to be more entrenched than something, hence to be more entrenched than $\perp$. Remembering that the Entailment condition (E2) must be respected by all E-relations, we know that all Cn_0-consequences of the $A_i \rightarrow B_i$'s must also be more entrenched than $\perp$. Abbreviating the "L_0-counterpart" $\{A_i \rightarrow B_i : A_i \Box\!\!\rightarrow B_i \in \Gamma\}$ of Γ by $L_0(\Gamma)$, we now know that all sentences in $\Delta_1 =_{df} Cn_0(L_0(\Gamma))$ obtain at least the first rank of epistemic entrenchment. In the second step, we collect all those $\neg A_j$'s which are in Δ_1. The corresponding inequalities $\neg A_j < A_j \rightarrow B_j$ are triggered and force all the $A_j \rightarrow B_j$'s to be more entrenched than the rest — except for the Cn_0-consequences of the $A_j \rightarrow B_j$'s, which are also lifted up to the second rank of epistemic entrenchment by (E2). This process of raising epistemic entrenchments as required by the "$\leq$-translations" of Γ and subsequent closing under Cn_0 is repeated time and again. In the limit, we take intersections. Roughly, we are ready if no inequality is triggered any longer. There may arise serious complications but they cannot be examined without a formal definition.

Definition 11 *Let* $\Gamma = \{A_i \Box\!\!\rightarrow B_i : i \in I\}$ *be a set of* L_1*-sentences. Then* $\leq_\Gamma = E(\Gamma)$ *is defined as follows. Put*

$$
\begin{array}{lcl}
\Delta_0 & = & L_0 \\
\Delta_{\alpha+1} & = & Cn_0(\{A_i{\to}B_i{:}\,\neg A_i{\in}\Delta_\alpha\}) \\
\Delta_\alpha & = & \bigcap\{\Delta_\beta{:}\,\beta{<}\alpha\} \quad \textit{for limit ordinals } \alpha
\end{array}
$$

and

$$
\alpha(\Gamma) = \begin{cases} \emptyset\ , & \textit{if } \Delta_\alpha = Cn_0(\emptyset) \textit{ and} \\ & \Delta_\beta = Cn_0(\emptyset) \textit{ for some } \beta < \alpha, \\ \Delta_\alpha\ , & \textit{if } \Delta_\alpha = Cn_0(\emptyset) \textit{ and} \\ & \Delta_\beta \neq Cn_0(\emptyset) \textit{ for all } \beta < \alpha, \\ \Delta_\alpha - \Delta_{\alpha+1}\ , & \textit{otherwise.} \end{cases}
$$

Then for every L_0-sentence C, $\mathrm{rank}_\Gamma(\mathrm{C}){=}\alpha$ *iff $C{\in}\alpha(\Gamma)$, and for every pair of L_0-sentences C and D,* $\mathrm{C}{\leq_\Gamma}\mathrm{D}$ *iff $rank_\Gamma(C){\leq}rank_\Gamma(D)$.*

A number of tasks lies before us. First, we have to check whether the definition makes sense at all, i.e., whether every L_0-sentence gets a unique rank number α. We shall see that the definition works fine and terminates after a finite number of steps if Γ is consistent and finite. It is no real disadvantage that it fails for inconsistent premise sets, but it will be interesting to observe in which of the infinite cases it fails. Secondly, we verify that in all successful cases, the definition actually generates an E-relation for Γ. Thirdly and lastly, we show that $\mathrm{E}(\Gamma)$ is the $\sqsubset_\mathrm{E}$-smallest E-relation for Γ.

The primary case in the definition of $\alpha(\Gamma)$ is of course captured by the last line. The worst thing that can happen in the construction process is that for some ordinal α, $\Delta_{\alpha+1}$ is identical with $\Delta_\alpha{\neq}\mathrm{Cn}_0(\emptyset)$. For that would mean that not only $\Delta_{\alpha+1} = \mathrm{Cn}_0(\{\mathrm{A}_i{\to}\mathrm{B}_i{:}\,\neg\mathrm{A}_i{\in}\Delta_\alpha\}) = \mathrm{Cn}_0(\{\mathrm{A}_i{\to}\mathrm{B}_i{:}\,\neg\mathrm{A}_i{\in}\Delta_{\alpha+1}\}) = \Delta_{\alpha+2}$, but, by the same argument, that $\Delta_\gamma{=}\Delta_\alpha$ for every $\gamma{>}\alpha$. As a consequence, $\gamma(\Gamma)$ would be empty for $\gamma{>}\alpha$, and the processing of the $\leq$-translations of the premises in Γ would be interrupted. Consider two examples for illustration.

Example 3 Let $\Gamma = \{\mathrm{A}\,\Box\!\!\to\mathrm{B}\&\mathrm{C}, \mathrm{B}\,\Box\!\!\to\mathrm{A}\&\neg\mathrm{C}\}$. The translation in terms of epistemic entrenchment is

$$\neg\mathrm{A}{<}\mathrm{A}{\to}(\mathrm{B}\&\mathrm{C}) \quad \text{and} \quad \neg\mathrm{B}{<}\mathrm{B}{\to}(\mathrm{A}\&\neg\mathrm{C}).$$

Now Δ_0 is L_0, and Δ_1 is $\mathrm{Cn}_0(\{\mathrm{A}{\to}(\mathrm{B}\&\mathrm{C}), \mathrm{B}{\to}(\mathrm{A}\&\neg\mathrm{C})\})$, but this again is L_0. So $\neg$A and $\neg$B are in Δ_1, so Δ_2 is again $\mathrm{Cn}_0(\{\mathrm{A}{\to}(\mathrm{B}\&\mathrm{C}),\mathrm{B}{\to}(\mathrm{A}\&\neg\mathrm{C})\} = \mathrm{L}_0$, and so on for every Δ_α. We never get an acceptable result.

Example 4 Another problematic case is $\Gamma = \{\mathrm{A}_i{\vee}\mathrm{A}_{i+1}\Box\!\!\to\neg\mathrm{A}_i{:}\,\mathrm{i}{=}1,2,3,\ldots\}$. The $\leq$-translations are

$$\neg A_i \& \neg A_{i+1} < (A_i \vee A_{i+1}) \rightarrow \neg A_i, \quad i=1,2,3,\ldots,$$

or equivalently,

$$\neg A_{i+1} < \neg A_i, \quad i=1,2,3,\ldots.$$[18]

As always, Δ_0 is L_0. Δ_1 is $Cn_0(\{(A_i \vee A_{i+1}) \rightarrow \neg A_i: i=1,2,3,\ldots\}) = Cn_0(\{\neg A_i: i=1,2,3,\ldots\})$. But then, for every $i=1,2,3,\ldots$, $\neg A_i \& \neg A_{i+1}$ is in Δ_1, so Δ_2 is again $Cn_0(\{\neg A_i: i=1,2,3,\ldots\})$, and the same for every Δ_α. We never manage to exploit the information provided by Γ. (End of examples)

It turns out that the two premise sets have a different status. In Example 3, Γ is inconsistent, and we shall see presently that every *finite* premise set which leads into this problem is inconsistent. So we need not bother about the problem for finite premise sets too much. In Example 4, on the other hand, Γ is consistent, since it is satisfied e.g. by the E-relation generated by the base

$$\bot \prec \ldots \prec \neg A_3 \prec \neg A_2 \prec \neg A_1 \prec \top\ .$$

The point illustrated by Example 4 is that there are premise sets which do not admit well-ordering E-relations. Since Γ translates to $\neg A_{i+1} < \neg A_i$, $i=1,2,3,\ldots$, it is clear that no E-relation $\leq$ for Γ can pick out an $\leq$-minimal sentence from the set $\{\neg A_i: i=1,2,3,\ldots\}$. But as our definition is made for well-ordering E-relations only, it is to be expected that it does not work fine in cases like Example 4. We suggest the following well-behavedness criterion for infinite premise sets:

Definition 12 *A premise set* $\Gamma = \{A_i \Box\!\!\rightarrow B_i: i \in I\}$ *is called* well-founded *iff it satisfies the condition*
$\{\neg A_j: j \in J\} \not\subseteq Cn_0(\{A_j \rightarrow B_j: j \in J\})$*, for every non-empty* $J \subseteq I$*.*

Observe that only well-founded premise sets Γ admit well-ordering E-relations for Γ. For assume Γ is not well-founded and $J \neq \emptyset$ is such that $\{\neg A_j: j \in J\} \subseteq Cn_0(\{A_j \rightarrow B_j: j \in J\})$. Suppose for reductio that $\leq$ is well-ordering and satisfies Γ. Consider $\{A_j \rightarrow B_j: j \in J\}$, and take a smallest element $A_k \rightarrow B_k$ of this set. By assumption, $\neg A_k \in Cn_0(\{A_j \rightarrow B_j: j \in J\})$. So, by (E2), $A_j \rightarrow B_j \leq \neg A_k$ for some $j \in J$. But $A_k \rightarrow B_k \leq A_j \rightarrow B_j$, so by (E1) $A_k \rightarrow B_k \leq \neg A_k$,

[18] Examples like this have been the subject of considerable discussion in the literature. Measure again Lewis's (1973, p. 20) line and instantiate A_i as 'Lewis's line is 1+(1/i) inches long.'

so $\leq$ fails to satisfy $A_k \Box\!\!\rightarrow B_k \in \Gamma$, so $\leq$ is no E-relation for Γ, and we have a contradiction.

Now let us carefully collect some basic facts concerning the construction of $E(\Gamma)$.

Observation 10 *Let $\Gamma = \{A_i \Box\!\!\rightarrow B_i : i \in I\}$ be a set of L_1-sentences. Then*

(i) for all α, $Cn_0(\emptyset) \subseteq \Delta_{\alpha+1} \subseteq \Delta_\alpha$;

(ii) for all ordinals α such that $\alpha(\Gamma) = \Delta_\alpha - \Delta_{\alpha+1}$, $\overleftarrow{\alpha}(\Gamma) =_{\mathrm{df}} \bigcup\{\beta(\Gamma) : \beta < \alpha\} = L_0 - \Delta_\alpha$;

(iii) if Γ is finite and consistent then Γ is well-founded.

Furthermore, if Γ is well-founded, then

(iv) for all α, if $\Delta_\alpha \subseteq \Delta_{\alpha+1}$ then $\Delta_\alpha = Cn_0(\emptyset)$;

(v) for all α, $\{A_i \rightarrow B_i : \neg A_i \in \Delta_{\alpha+1}\} \subseteq \{A_i \rightarrow B_i : \neg A_i \in \Delta_\alpha\}$, and if $\{A_i \rightarrow B_i : \neg A_i \in \Delta_{\alpha+1}\} = \{A_i \rightarrow B_i : \neg A_i \in \Delta_\alpha\}$, then $\{A_i \rightarrow B_i : \neg A_i \in \Delta_\alpha\} = \emptyset$;

(vi) there is an α such that $\top \in \alpha(\Gamma) = Cn_0(\emptyset)$; in particular, if Γ is finite and has n elements then $\top \in \alpha(\Gamma)$ for some $\alpha \leq n+1$;

(vii) for every L_0-sentence A, there is exactly one α such that $A \in \alpha(\Gamma)$;

(viii) for every α, Δ_α is an L_0-cut with respect to $\leq_\Gamma$, and for every non-empty L_0-cut S with respect to $\leq_\Gamma$, $S = \Delta_\alpha$ for some α;

(ix) for all L_0-sentences A and B, $A \leq_\Gamma B$ iff $A \in \Delta_\alpha$ implies $B \in \Delta_\alpha$ for every α.

Part (iii) of Observation 10 shows that we will have no problems if Γ is finite, and part (vi) shows that in this case the number of steps to be performed in the construction does not essentially exceed the number of conditionals in Γ. Part (vii) shows that rank_Γ is a function assigning to every L_0-sentence an ordinal. So $A \leq_\Gamma B$ iff $\mathrm{rank}_\Gamma(A) \leq \mathrm{rank}_\Gamma(B)$. We shall make use of this in the following. Parts (i) and (iv) make clear that this function is onto some initial segment $\{\beta : \beta < \alpha\}$ of the ordinals. All ranks are occupied. Parts (viii) and (ix) exhibit a similarity of the construction of an E-relation $E(\Gamma)$ from a given set of premises Γ with the construction of $E(\preceq)$ from a given E-base $\langle \mathcal{B}, \preceq \rangle$. In fact, the whole construction of Definition 11 may be viewed as the establishment of an E-base with $\mathcal{B} = L_0(\Gamma)$ and $\preceq\, = \,\leq_\Gamma \cap\, \mathcal{B} \times \mathcal{B}$. The $\neg A_i$'s just help us to determine the relations between the $A_i \rightarrow B_i$'s under minimalization. After these preparations, the following result will hardly be surprising.

Observation 11 *Let $\Gamma = \{A_i \Box\!\!\rightarrow B_i : i \in I\}$ be a well-founded set of L_1-sen-*

tences. Then

(i) $E(\Gamma)$ is a well-ordering E-relation.

(ii) $E(\Gamma)$ satisfies Γ, i.e., $\Gamma \subseteq K(E(\Gamma))$.

We see that what we have constructed is in fact an E-relation for Γ. To substantiate that we reach our final aim, we need a further technical lemma.

Observation 12 *Let $\Gamma = \{A_i \Box\!\!\rightarrow B_i : i \in I\}$ be a well-founded set of L_1-sentences. Then for every L_0-sentence A, $rank_{E(\Gamma)}(A) = rank_\Gamma(A)$.*

Now we can prove what we have been after in this section.

Observation 13 *Let $\Gamma = \{A_i \Box\!\!\rightarrow B_i : i \in I\}$ be a well-founded set of L_1-sentences. Then for every E-relation $\leq$ satisfying Γ, $E(\Gamma) \sqsubset_E \leq$ or $\leq = E(\Gamma)$.*

Corollary *For well-ordering E-relations $\leq$, $E(K(\leq)) = \leq$.*

7 The logic of E-minimality

We propose to use $Cn_{\sqsubset_E}$ as the right consequence relation for conditionals.

Definition 13 *Let Γ be a set of L_1-sentences and A an L_1-sentence. Then $\Gamma \models A$, or equivalently $A \in Cn(\Gamma)$, iff $\Gamma \models_{\sqsubset_E} A$. Furthermore, $\Gamma \models\!\!\equiv A$ iff $\Gamma \models\!\!\equiv_{\sqsubset_E} A$.*

Recall that we just instantiate here the scheme of preferential entailment in the sense of Makinson (1989; 1990), with the underlying preferential model structure being $(\mathcal{E}, \models, \sqsubset_E)$ where $\mathcal{E}$ is the set of all E-relations over L_0 and $\models$ and $\sqsubset_E$ are as determined in Definitions 4 and 10. We allow infinite sets of premises, which will often give rise to infinite ranks of epistemic entrenchment. It is an effect of having done the whole thing for ordinals rather than for natural numbers that we can apply Cn to infinite sets of sentences. As usual, we may say that a set K of L_1-sentences is a *theory* or a *belief set* in L_1 if K=Cn(K).

In the well-behaved — i.e., well-founded — cases, $\sqsubset_E$ will perform interesting comparisons. But there are non-well-founded premise sets Γ which do not permit well-ordering E-relations, for instance that of Example 4, viz., $\Gamma = \{A_i \lor A_{i+1} \Box\!\!\rightarrow \neg A_i : i = 1,2,3,\ldots\}$. In this case, $\Gamma \models A$ coincides with $\Gamma \models_1 A$. On the other hand, recall that by the definition of $\sqsubset_E$, if there is one well-ordering E-relation $\leq$ for Γ then we need not consider any E-relations for Γ that are not well-ordering, because $\leq$ is $\sqsubset_E$-preferred to all of these. Moreover, if Γ is well-founded we know that there are well-ordering E-relations for Γ and that there is a $\sqsubset_E$-smallest among them, viz., $E(\Gamma)$. So for a well-founded Γ, $\Gamma \models A$ iff $E(\Gamma) \models A$, or in other words, $Cn(\Gamma)=K(E(\Gamma))$. It is no problem to determine the consequences of many suspicuous-looking infinite premise sets like e.g. $\Gamma=\{A_i \vee A_{i+1} \Box\!\!\rightarrow \neg A_{i+1}: i=1,2,3,\ldots\}$.

Our consequence relation accounts for the paradigm example presented in the introduction and Example 1 in a satisfactory and almost trivial way. If all we know is given by $\Gamma=\{A \vee B\}$, then the unique $\sqsubset_E$-minimal E-relation for Γ is given by the E-base $\bot \prec A \vee B \prec \top$, which satisfies $\neg A \Box\!\!\rightarrow B$. But if we then learn that A (and $\neg$B) then all we know is $\Gamma=\{A \vee B, A(,\neg B)\}$, so the unique $\sqsubset_E$-minimal E-relation is given by $\bot \prec A \vee B \simeq A (\simeq \neg B) \prec \top$ which fails to satisfy $\neg A \Box\!\!\rightarrow B$. That is, $Cn_{\sqsubset_E}$ in fact validates (3) and invalidates (4) mentioned in the discussion of Example 1. It is obvious but notable that Cn is *nonmonotonic* and allows the *inference of conditionals from non-conditional knowledge bases.*

In Example 2, the second E-base is discarded, so we get for instance that $\Gamma=\{A, B, \neg A \Box\!\!\rightarrow B\}$ entails the contraposed conditional $\neg B \Box\!\!\rightarrow A$.

Besides the performance of Cn in examples, its abstract properties are of interest.[19]

Observation 14 *Cn satisfies*

(i) $Cn_0(\Gamma) \cap L_1 \subseteq Cn(\Gamma) \cap L_1$ *(Restricted Supraclassicality)*

(ii) for all L_0*-sentences A and B,* $\Gamma \cup \{A\} \models B$ *iff* $\Gamma \models A \rightarrow B$ *(Restricted Deduction Theorem)*

(iii) $\Gamma \subseteq Cn(\Gamma)$ *(Inclusion)*

(iv) $Cn(Cn(\Gamma)) = Cn(\Gamma)$ *(Idempotence)*

(v) if $\Gamma \subseteq \Sigma \subseteq Cn(\Gamma)$ *then* $Cn(\Sigma) \subseteq Cn(\Gamma)$ *(Cut)*

(vi) if $\Gamma \subseteq \Sigma \subseteq Cn(\Gamma)$ *then* $Cn(\Gamma) \subseteq Cn(\Sigma)$ *(Cautious Monotony)*

(vii) if $\Gamma_2 \subseteq Cn(\Gamma_1)$, $\Gamma_3 \subseteq Cn(\Gamma_2), \ldots, \Gamma_n \subseteq Cn(\Gamma_{n-1})$, $\Gamma_1 \subseteq Cn(\Gamma_n)$ *then* $\Gamma_i = \Gamma_j$ *for every* $i,j \leq n$ *(Loop)*

Parts (iii)–(vi) mean that Cn is a *cumulative* inference relation in Makinson's sense. We point out that the restriction of Supraclassicality is severe. For example, if we have $\Gamma \models A \Box\!\!\rightarrow B$ and $\Gamma \models C \Box\!\!\rightarrow D$, we would certainly like to take over the classical step to $\Gamma \models (A \Box\!\!\rightarrow B) \& (C \Box\!\!\rightarrow D)$, but this already transcends the bounds of the language L_1. A similar restriction applies to the Deduction

[19]The names of the conditions to be discussed are taken from Makinson(1989; 1990).

Theorem. In Part 2 of the present trilogy we shall extend L_1 in order to attain full Supraclassicality and give a treatment of negations and disjunctions of conditionals as well.

Observation 14 shows where Cn behaves well. But there are also less mannerly features.

Observation 15 *Cn does not satisfy*

(i) *if* $\Gamma\cup\{A\}\models C$ *and* $\Gamma\cup\{B\}\models C$ *then* $\Gamma\cup\{A\vee B\}\models C$ *(Disjunction in the Antecedent)*

(ii) *if* $\Gamma\cup\{A\}\models B$ *and* $\Gamma\cup\{\neg A\}\models B$ *then* $\Gamma\models B$ *(Proof by Cases)*

(iii) *if* $\Gamma\models B$ *then* $\Gamma\cup\{A\}\models B$ *or* $\Gamma\cup\{\neg A\}\models B$ *(Negation Rationality)*

(iv) *if* $\{A,B\}\not\models\bot$ *then* $Cn(\{A\})\cup\{B\}\not\models\bot$ *(Consistency Preservation)*

I feel that I should give an example that makes the case for at least one of these results. Let me explain how Disjunction in the Antecedent can fail.

Example 5 We consider another restaurant example and assume now that there is a third restaurant which is run by Debbie. Let $\Gamma=\{\neg A\Box\!\!\rightarrow B\vee D, \neg B\Box\!\!\rightarrow A\vee D\}$. Now suppose you (only) know that either Annie's or Ben's restaurant is open. In this situation, Γ adds nothing new, since the information provided by

> If Annie's restaurant is not open (then) Ben's or Debbie's restaurant will be open.

and

> If Ben's restaurant is not open (then) Annie's or Debbie's restaurant will be open.

is already contained in the information provided by $A\vee B$. But suppose you (only) know that Annie's restaurant is open. In this case the first element in Γ, now read as

> If Annie's restaurant were not open (then) Ben's or Debbie's restaurant would be open.

does contain additional information. In particular, it seems justified to infer from $\Gamma\cup\{A\}$ the following conditional:

> If neither Annie's nor Ben's restaurant were open (then) Debbie's restaurant would be open. (5)

By an analogous argument, we can infer (5) from the premise set $\Gamma\cup\{B\}$. From $\Gamma\cup\{A\vee B\}$, on the other hand, we saw that we cannot get anything over and above the conclusions which can be drawn from $\{A\vee B\}$ alone, so in particular we cannot get (5).[20] (End of example)

The failure of Disjunction in the Antecedents and Proof by Cases is neither very common nor very uncommon in nonmonotonic logics. The failure of Negation Rationality which is common in nonmonotonic logics implies the failure of more non-Horn conditions for Cn (see Makinson 1990, Section IV.1). The failure of Consistency Preservation is perhaps the most striking deviation from usual patterns. In fact, the proof shows that for arbitrary independent L_0-sentences A and B, Cn({A}) and Cn({B}) are not satisfiable simultaneously. This already indicates that the addition of new items of belief should not be performed by taking the logical expansion of the current theory, but by generating a new theory from the augmented premise set. Belief change based on Cn will be the topic of Part 3 of the trilogy.

For reasons of language restriction, we cannot directly compare our logic with Lewis's "official" logic **VC**. With the plausible consistency condition for L_2

$$\text{if } \leq\ \models \neg(A\Box\!\!\rightarrow B) \quad \text{then} \quad \leq\ \not\models A\Box\!\!\rightarrow B\ ,$$

however, we get the following translations of the prominent **VC**-axioms into robust inferences. We refer to the axiomatization of **VC** given by Gärdenfors (1988, Section 7.2).

Observation 16 *Let A,B,C range over L_0-sentences. Then the following sentence schemes of the form $(A_0\&\ldots\&A_n)\rightarrow B$ ($n\geq 0$), which are axioms for* **VC**, *are translatable into valid robust inferences of the form $\{A_0,\ldots,A_n\}\models\!\!\!\equiv B$:*

(i) A, *for* $A\in Cn_0(\emptyset)$;

(ii) $((A\Box\!\!\rightarrow B)\&(A\Box\!\!\rightarrow C)) \rightarrow (A\Box\!\!\rightarrow B\&C)$;

(iii) $A\Box\!\!\rightarrow\top$;

(iv) $A\Box\!\!\rightarrow A$;

(v) $(A\Box\!\!\rightarrow B) \rightarrow (A\rightarrow B)$;

(vi) $(A\&B) \rightarrow (A\Box\!\!\rightarrow B)$;

[20] For details, see the proof of Observation 15(i). — Notice that the switch from indicative to subjunctive mood seems to produce some change in the meaning of the conditionals in question. Both types of conditionals (if they constitute any clear-cut types at all) are covered by our analysis.

(vii) $(A\Box\!\!\rightarrow\neg A) \rightarrow (B\Box\!\!\rightarrow\neg A)$;
(viii) $((A\Box\!\!\rightarrow B)\&(B\Box\!\!\rightarrow A)\&(A\Box\!\!\rightarrow C)) \rightarrow (B\Box\!\!\rightarrow C)$;
(ix) $((A\Box\!\!\rightarrow C)\&(B\Box\!\!\rightarrow C)) \rightarrow (A\vee B\Box\!\!\rightarrow C)$;
(x) $((A\Box\!\!\rightarrow C)\&\neg(A\Box\!\!\rightarrow\neg B)) \rightarrow (A\&B\Box\!\!\rightarrow C)$.

Furthermore, the **VC**-*rule "from* $B\rightarrow C$, *to infer* $(A\Box\!\!\rightarrow B)\rightarrow(A\Box\!\!\rightarrow C)$*" is translatable into the valid robust inference*

(xi) if $C\in Cn_0(B)$ *then* $A\Box\!\!\rightarrow B \models\!\!\equiv A\Box\!\!\rightarrow C$.

When pressed to name the most distinctive feature of natural language conditionals as opposed to "conditionals" encountered in logic and mathematics, I think the best thing one can do is point out that natural language conditionals fail to satisfy some cherished inference patterns. There is a canon of three arguments which have become known as the *counterfactual fallacies* (see Lewis 1973, Section 1.8):

Strengthening the Antecedent (SA)	*Transitivity (Tr)*	*Contraposition (Cp)*
$\dfrac{A\Box\!\!\rightarrow B}{A\&C\Box\!\!\rightarrow B}$	$\dfrac{A\Box\!\!\rightarrow B \quad B\Box\!\!\rightarrow C}{A\Box\!\!\rightarrow C}$	$\dfrac{A\Box\!\!\rightarrow B}{\neg B\Box\!\!\rightarrow\neg A}$

Most writers agree that these schemes are not universally valid for conditionals. Yet there seem to be many contexts in which one may safely make use of them. The present logic accounts for this fact by construing conditionals in such a way that (SA), (Tr) and (Cp) are *valid by default*. That is, the relevant premises taken in isolation entail the respective conclusions, but the inference is not robust, since it can be spoilt by augmenting the premise set. We can give a precise description of the contexts in which the so-called counterfactual fallacies fail.

Observation 17

(i) $\{A\Box\!\!\rightarrow B\}\models A\&C\Box\!\!\rightarrow B$, *but not* $\{A\Box\!\!\rightarrow B\}\models\!\!\equiv A\&C\Box\!\!\rightarrow B$;
$\{A\Box\!\!\rightarrow B,\ B\Box\!\!\rightarrow C\}\models A\Box\!\!\rightarrow C$, *but not* $\{A\Box\!\!\rightarrow B, B\Box\!\!\rightarrow C\}\models\!\!\equiv A\Box\!\!\rightarrow C$;
$\{A\Box\!\!\rightarrow B\}\models\neg B\Box\!\!\rightarrow\neg A$, *but not* $\{A\Box\!\!\rightarrow B\}\models\!\!\equiv\neg B\Box\!\!\rightarrow\neg A$.

(ii) In particular, $\{A\Box\!\!\rightarrow B,\ A\Box\!\!\rightarrow\neg C\}\not\models A\&C\Box\!\!\rightarrow B$;
$\{A\Box\!\!\rightarrow B,\ B\Box\!\!\rightarrow C,\ B\Box\!\!\rightarrow\neg A\}\not\models A\Box\!\!\rightarrow C$;
$\{A\Box\!\!\rightarrow B,\ B\}\not\models\neg B\Box\!\!\rightarrow\neg A$.

(iii) For every E-relation $\leq$, *if* $\leq$ *satisfies the premise but not the conclusion*

of (SA) then $\leq$ satisfies $A\Box\!\!\rightarrow\neg C$; if $\leq$ satisfies the premises but not the conclusion of (Tr) then $\leq$ satisfies $B\Box\!\!\rightarrow\neg A$; if $\leq$ satisfies the premise but not the conclusion of (Cp) then $\leq$ satisfies B.

Some comments are in order. In order to have a proper understanding of what is "counterfactual" in these inference schemes, we should introduce the appropriate terms in our setting.

Definition 14 *Let $\leq$ be an E-relation and A and B L_0-sentences. Then the conditional $A\Box\!\!\rightarrow B$ is called* open *(with respect to $\leq$) if $A\leq\bot$ and $\neg A\leq\bot$; it is called* (weakly) counterfactual *(with respect to $\leq$) if $\bot<\neg A$ and* strongly counterfactual *(with respect to $\leq$) if $\bot<\neg A$ and $\bot<\neg B$; it is called* factual *(with respect to $\leq$) if $\bot<A$; it is called* even if type *(with respect to $\leq$) if $\bot<B$.*

The positive parts of Observation 17(i), are, as they stand, about open conditionals. But it is easy to verify that they are extendable to counterfactual and strongly counterfactual conditionals, in the sense that the pertinent conditions of Definition 14 are added to the premise set. Note, however, the exceptional status of (Cp). If both the premise and the conclusion of (Cp) are to be counterfactual, then they are also even if type. If the premise is to be strongly counterfactual, then the conclusion is a factual even if conditional.

Parts (ii) and (iii) of Observation 17 are more interesting. Considering the proof, we discover that (SA), (Tr) and (Cp) only fail if counterfactual conditionals are involved. As to (SA), the conclusion must be counterfactual; as to (Tr), the first premise and the conclusion must be counterfactual, and, by the same token, $B\Box\!\!\rightarrow\neg A$ is even if type; as to (Cp), again the conclusion must be counterfactual while the premise is even if type. So there is no failure of the "counterfactual fallacies" if all conditionals involved are open. This is what justifies the predicate 'counterfactual'. The predicate 'fallacy' is, as Part (i) of Observation 17 shows, not quite appropriate.

Appendix: Proofs of Observations

Proof of Observation 1 (i) The limiting case $\vdash\neg A$ is immediate. So let $\nvdash\neg A$; then, by the definition of $*_K = R(C(\leq_K))$, an L_0-sentence C is in $K*_A$ iff it is in $Cn_0((K\cap\{B: \neg A<_K\neg A\vee B\})\cup\{A\})$, i.e., iff $A\to C\in K\cap\{B: \neg A<_K\neg A\vee B\}$, i.e., by (E4), iff $\bot<_K A\to C$ and $\neg A<_K\neg A\vee(A\to C)$, i.e., by the properties of E-relations, iff $\neg A<_K\neg A\vee C$, i.e., iff C is in $\{B: \neg A<_K\neg A\vee B\}$.

(ii) By the definition of $\leq_K$, $A\leq_K B$ iff $A\notin(K\cap K*_{\neg(A\&B)})$ or $\vdash A\&B$. It remains to be shown that in the case where $\nvdash A\&B$, $A\notin K$ implies $A\notin K*_{\neg A\vee\neg B}$. Now if $A\notin K$, then, by the Gärdenfors postulates for revisions of belief sets in L_0, $K*_{\neg A\vee\neg B}=Cn_0(K\cup\{\neg A\vee\neg B\})$. But since belief sets are closed under Cn_0, $A\notin K$ is equivalent to $A\notin Cn_0(K)$, which in turn is equivalent to $A\notin Cn_0(K\cup\{\neg A\vee\neg B\}) = K*_{\neg A\vee\neg B}$, so we are done.

(iii) Immediate from the results in Gärdenfors (1988, Section 3.6) and Gärdenfors and Makinson (1988). □

Proof of Observation 2 Let $\leq=E(\preceq)$. Transitivity (E1) is trivial. — For Entailment (E2), assume that $\emptyset\neq\Gamma\vdash A$. We have to show that $B\leq A$ for some $B\in\Gamma$. Since classical propositional logic is compact, there is a finite $\Gamma_0\subseteq\Gamma$ such that $\Gamma_0\vdash A$. Now suppose for reductio that $B\not\leq A$ for every $B\in\Gamma$. Hence $B\not\leq A$ for every $B\in\Gamma_0$, i.e., by the definition of $E(\preceq)$, there is a $\mathcal{B}$-cut S_B for every $B\in\Gamma_0$ such that $S_B\vdash B$ but $S_B\nvdash A$. Consider $\bigcup\{Cn_0(S_B): B\in\Gamma_0\}$. Clearly, Γ_0 is included in $\bigcup\{Cn_0(S_B): B\in\Gamma_0\}$. Since $\mathcal{B}$-cuts are nested, the $Cn_0(S_B)$'s are nested, so, since Γ_0 is finite, $\bigcup\{Cn_0(S_B): B\in\Gamma_0\}$ is identical with $Cn_0(S_B)$ for some $B\in\Gamma_0$. For this B then, we have $\Gamma_0\subseteq Cn_0(S_B)$ and $S_B\nvdash A$. But this contradicts $\Gamma_0\vdash A$. — For Maximality (E3), assume that $B\leq A$ for every B. By definition, this means that for every B and every $\mathcal{B}$-cut S, if $B\in Cn_0(S)$ then $A\in Cn_0(S)$. Choose $B=\top$ and $S=\emptyset$. This gives us $\vdash A$, as desired. □

Proof of Observation 3 It is clear that (E2) for $\preceq$ is a necessary condition for $E(\preceq)\cap\mathcal{B}\times\mathcal{B}=\preceq$. For otherwise $E(\preceq)$ could not be an E-relation, in contradiction to Observation 2. To show conversely that (E2) for $\preceq$ is sufficient, assume that $\preceq$ satisfies (E2) over $\mathcal{B}$. We have to show that for all A and B in $\mathcal{B}$, $A\preceq B$ iff $A\in Cn_0(S)$ entails $B\in Cn_0(S)$ for all $\mathcal{B}$-cuts S. The direction from left to right follows immediately from the definition of a $\mathcal{B}$-cut and the monotonicity of Cn_0. For the direction from right to left, assume that $A\not\preceq B$. It remains to show that there is a $\mathcal{B}$-cut S such that $A\in Cn_0(S)$

but $B \notin Cn_0(S)$. Now consider $S_A =_{df} \{C \in \mathcal{B}: A \preceq C\}$. S_A is a $\mathcal{B}$-cut, since $\preceq$ is transitive. $A \in S_A$, since $\preceq$ is reflexive, so $S_A \neq \emptyset$ and $A \in Cn_0(S_A)$. $A \not\preceq B$ by assumption, so, since $\preceq$ is transitive, $C \not\preceq B$ for every C in S_A. Hence, by (E2) for $\preceq$ over $\mathcal{B}$, $B \notin Cn_0(S_A)$, and we are done. □

Proof of Observation 4 (i) Let $\Gamma \models_1 A$, i.e., every E-relation satisfying Γ satisfies A. This implies that every $\sqsubset_E$-minimal E-relation for some superset Σ of Γ satisfies A, i.e., $\Gamma \models_{\sqsubset} A$.

(ii) Let $\Gamma \models_{\sqsubset} A$. Assume that $\Gamma \not\models_1 A$, i.e., there is an E-relation $\leq$ which satisfies Γ without satisfying A. That is, $\Gamma \subseteq K(\leq)$ and $A \notin K(\leq)$. But by hypothesis, every Σ such that $\Gamma \subseteq \Sigma$ minimally implies A. So in particular every minimal E-relation for $K(\leq)$ must satisfy A. But since $\leq$ does not satisfy A, it cannot be minimal for $K(\leq)$. Therefore, the antecedent of (ii) is false. □

Proof of Observation 5 (i)$\Leftrightarrow$(ii): We have to show that $K(\leq) \subseteq K(\leq')$ iff $\leq' \subseteq \leq$. As remarked above, L_0-sentences A are satisfaction-equivalent to conditionals $\top \Box\!\!\rightarrow A$, so $K(\leq)$ and $K(\leq')$ can be thought of as consisting of conditionals only. We have to show that for all L_0-sentences A and B,

if $\neg A < \neg A \vee B$ then $\neg A <' \neg A \vee B$,

i.e., by the connectivity of E-relations,

if not $\neg A \vee B \leq \neg A$ then not $\neg A \vee B \leq' \neg A$, i.e.,

(*) if $\neg A \vee B \leq' \neg A$ then $\neg A \vee B \leq \neg A$,

iff for all L_0-sentences C and D,

(**) if $C \leq' D$ then $C \leq D$.

The direction from (**) to (*) is immediate. To see that the converse also holds, substitute $\neg(C\&D)$ for A and C for B in (*). This gives us

if $\neg\neg(C\&D) \vee C \leq' \neg\neg(C\&D)$ then $\neg(C\&D) \vee C \leq \neg\neg(C\&D)$,

i.e., by (E2),

if $C \leq' C\&D$ then $C \leq C\&D$.

But since for every E-relation $\leq$, $C \leq C\&D$ is equivalent to $C \leq D$, the latter condition is equivalent to (**).

(iii)$\Leftrightarrow$(ii): This is immediate from the connectivity of E-relations which gives us $< = (L_0 \times L_0) - \leq$ and $<' = (L_0 \times L_0) - \leq'$.

(iv)$\Leftrightarrow$(ii): That $\leq \subseteq \leq'$ implies $\doteq \subseteq \doteq'$ is clear from the definitions $\doteq = \leq \cap \leq^{-1}$ and $\doteq' = \leq' \cap (\leq')^{-1}$. To verify the converse, assume that $\leq \not\subseteq \leq'$, i.e., that there are L_0-sentences A and B such that $A \leq B$ but not $A \leq' B$. By the properties of E- relations, this implies that $A \doteq A\&B$ but not $A \doteq' A\&B$, so $\doteq \not\subseteq \doteq'$, and we are done. □

Proof of Observation 6 By a number of Boolean transformations of the definition of $\sqsubset_{K_0}$. □

Proof of Observation 7 (i) As to conditionals, we show that for every B and C in L_0,

$\neg B\vee C\not\leq\neg B$ iff $\neg B\vee C\not\leq_i\neg B$ for all i, i.e.,

$\neg B\vee C\leq\neg B$ iff $\neg B\vee C\leq_i\neg B$ for some i.

But this immediate from the definition of $\leq$. The case of L_0-sentences is similar.

(ii) Let $\leq^*$ be as indicated. Then, just as before, for every B and C in L_0,

$\neg B\vee C\leq^*\neg B$ iff $\neg B\vee C\leq_i\neg B$ for some i.

Substituting $\neg(D\&E)$ for B and D for C gives us

$\neg\neg(D\&E)\vee D\leq^*\neg\neg(D\&E)$ iff $\neg\neg(D\&E)\vee D\leq_i\neg\neg(D\&E)$ for some i, i.e., as all relations involved are E-relations,

$D\leq^*D\&E$ iff $D\leq_i D\&E$ for some i.

But again, since all relations involved are E-relations, this is equivalent to

$D\leq^*E$ iff $D\leq_i E$ for some i,

i.e., since D and E were chosen arbitrarily, $\leq^* = \leq_1\cup\ldots\cup\leq_n = \leq$, as desired.

(iii) Clearly, any union of E-relations satisfies Entailment (E2) and Maximality (E3). So $\leq=\leq_1\cup\ldots\cup\leq_n$ is an E-relation iff it satisfies Transitivity (E1), i.e., iff for all $A,B,C\in L_0$,

if $B\leq_i C$ for some i and $C\leq_j A$ for some j, then $B\leq_k A$ for some k.

By a simple Boolean transformation, we see that this is violated iff $A<_i B\leq_i C$ for some i and $C\leq_j A<_j B$ for some j, and $A<_k B$ for every k, as desired. □

Proof of Observation 8 Let $\leq\sqsubseteq_K\leq'$. From Observation 5 we know that this is equivalent to $\leq'\subseteq\leq$.

We first show by transfinite induction on α that for any ordinal α, $\overleftarrow{\alpha}(\leq')\subseteq\overleftarrow{\alpha}(\leq)$.

- $\overleftarrow{0}(\leq')\subseteq\overleftarrow{0}(\leq)$: $\overleftarrow{0}(\leq') = \emptyset = \overleftarrow{0}(\leq)$.
- $\overleftarrow{\alpha+1}(\leq')\subseteq\overleftarrow{\alpha+1}(\leq)$: The induction hypothesis is $\overleftarrow{\beta}(\leq')\subseteq\overleftarrow{\beta}(\leq)$ for every $\beta<\alpha+1$. So in particular $\overleftarrow{\alpha}(\leq')\subseteq\overleftarrow{\alpha}(\leq)\subseteq\overleftarrow{\alpha+1}(\leq)$. As $\overleftarrow{\alpha+1}(\leq') = \overleftarrow{\alpha}(\leq')\cup \alpha(\leq')$, it remains to show that $\alpha(\leq')\subseteq\overleftarrow{\alpha+1}(\leq)$:

$$\begin{aligned}
&\alpha(\leq') = \\
= \ &\{B\in L_0-\overleftarrow{\alpha}(\leq')\colon B\leq' C \text{ for every } C\in L_0-\overleftarrow{\alpha}(\leq')\} = \\
= \ &(\{B\in L_0-\overleftarrow{\alpha}(\leq')\colon B\leq' C \text{ for every } C\in L_0-\overleftarrow{\alpha}(\leq')\} \cap \overleftarrow{\alpha}(\leq)) \cup \\
&(\{B\in L_0-\overleftarrow{\alpha}(\leq')\colon B\leq' C \text{ for every } C\in L_0-\overleftarrow{\alpha}(\leq')\} \cap L_0-\overleftarrow{\alpha}(\leq)) \subseteq
\end{aligned}$$

$\subseteq\ \overleftarrow{\alpha}(\leq) \cup \{B{\in}(L_0{-}\overleftarrow{\alpha}(\leq')){\cap}(L_0{-}\overleftarrow{\alpha}(\leq)) : B{\leq'}C$ for every $C{\in}L_0{-}\overleftarrow{\alpha}(\leq')\} \subseteq$ (by the induction hypothesis)

$\subseteq\ \overleftarrow{\alpha}(\leq) \cup \{B{\in}L_0{-}\overleftarrow{\alpha}(\leq) : B{\leq'}C$ for every $C{\in}L_0{-}\overleftarrow{\alpha}(\leq)\} \subseteq$ (by $\leq'{\subseteq}\leq$)

$\subseteq\ \overleftarrow{\alpha}(\leq) \cup \{B{\in}L_0{-}\overleftarrow{\alpha}(\leq) : B{\leq}C$ for every $C{\in}L_0{-}\overleftarrow{\alpha}(\leq)\} =$

$=\ \overleftarrow{\alpha}(\leq) \cup \alpha(\leq) =$

$=\ \overleftarrow{\alpha{+}1}(\leq)$.

• $\overleftarrow{\alpha}(\leq'){\subseteq}\overleftarrow{\alpha}(\leq)$ for limit ordinals α: The induction hypothesis is $\overleftarrow{\beta}(\leq'){\subseteq}\overleftarrow{\beta}(\leq)$ for every $\beta{<}\alpha$. But since $\overleftarrow{\alpha}(\leq') = \bigcup\ \{\beta(\leq'){:}\beta{<}\alpha\} = \bigcup\{\overleftarrow{\beta{+}1}(\leq'){:}\beta{<}\alpha\}$, and similarly for $\leq$, we get the claim immediately from the induction hypothesis. Having shown that for any ordinal α, $\overleftarrow{\alpha}(\leq'){\subseteq}\overleftarrow{\alpha}(\leq)$, we can rerun the argument establishing $\alpha(\leq'){\subseteq}\overleftarrow{\alpha{+}1}(\leq)$, this time for any ordinal α. But this just means that for every L_0-sentence A, if $\mathrm{rank}_{\leq'}{=}\alpha$ then $\mathrm{rank}_{\leq}{\leq}\alpha$, and we are done. □

Proof of the Corollary Immediate from the Observation and the fact $\leq{\sqsubseteq_E}\leq'$ and $\leq'{\sqsubseteq_E}\leq$ implies $\leq{=}\leq'$. □

Proof of Observation 9 Let $\leq{\sqsubseteq_E}\leq'$. If also $\leq'{\sqsubseteq_E}\leq$, then, by the antisymmetry of $\sqsubseteq_E$, $\leq{=}\leq'$, so $\leq{\sqsubseteq_{K_0}}\leq'$ is trivial. Now consider the principal case where $\leq'{\not\sqsubseteq_E}\leq$. By Observation 8, this gives us $\leq'{\not\sqsubseteq_K}\leq$. Hence either $\leq{\sqsubseteq_K}\leq'$ or $\leq$ and $\leq'$ are incomparable with respect to $\sqsubseteq_K$. In the former case, $\leq{\sqsubseteq_{K_0}}\leq'$ is immediate. In the latter case, we have to show the $K_0(\leq){\subseteq}K_0(\leq')$. But this just means that $L_0{-}0(\leq){\subseteq}L_0{-}0(\leq')$ which is entailed by $\leq{\sqsubseteq_E}\leq'$. □

Proof of Observation 10 (i) As Cn_0 is monotonic, $Cn_0(\emptyset)$ is included in Δ_α for every α. $\Delta_{\alpha+1} = Cn_0(\{A_j{\to}B_j{:}\,\neg B_j{\in}\Delta_\alpha\}) \subseteq Cn_0(\{\neg B_j{:}\,\neg B_j{\in}\Delta_\alpha\}) \subseteq Cn_0(\Delta_\alpha) = \Delta_\alpha$, since Δ_α is closed under Cn_0.

(ii) By definition, $\overleftarrow{\alpha}(\leq){=}\bigcup\{\beta(\Gamma){:}\,\beta{<}\alpha\} = \bigcup\{\Delta_\beta{-}\Delta_{\beta+1}{:}\,\beta{<}\alpha\}$. We show by transfinite induction that the latter set equals $L_0{-}\Delta_\alpha$.

$\alpha{=}0$: $\bigcup\{\Delta_\beta{-}\Delta_{\beta+1}{:}\,\beta{<}0\} = \emptyset = L_0{-}\Delta_0$.

$\alpha{+}1$: $\bigcup\{\Delta_\beta{-}\Delta_{\beta+1}{:}\,\beta{<}\alpha{+}1\} = (\bigcup\{\Delta_\beta{-}\Delta_{\beta+1}{:}\,\beta{<}\alpha\}) \cup (\Delta_\alpha{-}\Delta_{\alpha+1}) =$ (by induction hypothesis) $(L_0{-}\Delta_\alpha){-}(\Delta_\alpha{-}\Delta_{\alpha+1}) = L_0{-}\Delta_{\alpha+1}$.

Limit ordinals α: $\bigcup\{\Delta_\beta{-}\Delta_{\beta+1}{:}\,\beta{<}\alpha\} =$ (by set theory) $\bigcup\{\bigcup\{\Delta_\gamma{-}\Delta_{\gamma+1}{:}\,\gamma{<}\beta\}{:}\,\beta{<}\alpha\} =$ (by induction hypothesis) $\bigcup\{L_0{-}\Delta_\beta{:}\,\beta{<}\alpha\} = L_0{-}\bigcap\{\Delta_\beta{:}\,\beta{<}\alpha\} =$ (by definition) $L_0{-}\Delta_\alpha$.

(iii) Let $\Gamma{=}\{A_i{\Box\!\!\to}B_i{:}\,i{\in}I\}$ be finite and consistent, and let $\leq$ be an E-relation for Γ. Now assume for reductio that there is a non-empty $J{\subseteq}I$ such that $\{\neg A_j : j\in J\}$ is contained in $Cn_0(\{A_j \to B_j : j\in J\})$. Observe that J is finite. So $\{A_j \to B_j : j\in J\} \vdash \&\{\neg A_j : j\in J\}$, where $\&\{\neg A_j : j\in J\}$ is the conjunction of the elements of $\{\neg A_j : j\in J\}$. Thus, by (E2), $A_k \to B_k \leq \&\{\neg A_j : j\in J\} \leq \neg A_k$ for some k in J. That is $\neg A_k \not< A_k \to B_k$, but this means that $\leq$ does not satisfy $A_k \Box\!\!\to B_k \in \Gamma$, contradicting our assumption that $\leq$ is an E-relation for Γ.

For the following, assume that Γ is well-founded.

(iv) Let α be such that $\Delta_\alpha \subseteq \Delta_{\alpha+1}$. Hence, by (i), $\Delta_\alpha = \Delta_{\alpha+1}$. By definition then, $\Delta_{\alpha+1} = Cn_0(\{A_j \to B_j : \neg A_j \in \Delta_\alpha\}) = Cn_0(\{A_j \to B_j : \neg A_j \in \Delta_{\alpha+1}\})$. Set $J = \{j\in I : \neg A_j \in \Delta_{\alpha+1}\}$. Then $\{\neg A_j : j\in J\} = \{\neg A_j : \neg A_j \in \Delta_{\alpha+1}\} \subseteq \Delta_{\alpha+1} = Cn_0(\{A_j \to B_j : \neg A_j \in \Delta_{\alpha+1}\}) = Cn_0(\{A_j \to B_j : j\in J\})$. Hence, by the well-foundedness of Γ, $J = \{j\in I : \neg A_j \in \Delta_{\alpha+1}\} = \emptyset$. That is, since $\Delta_\alpha = \Delta_{\alpha+1}$, $\{j\in I : \neg A_j \in \Delta_\alpha\} = \emptyset$, i.e., by definition, $\Delta_{\alpha+1} = Cn_0(\emptyset)$, hence, by $\Delta_\alpha = \Delta_{\alpha+1}$ again, $\Delta_\alpha = Cn_0(\emptyset)$.

(v) The first part of (v) follows immediately from (i). For the second part, suppose that $\{A_i \to B_i : \neg A_i \in \Delta_{\alpha+1}\} = \{A_i \to B_i : \neg A_i \in \Delta_\alpha\}$. So, by definition, $\Delta_{\alpha+2} = \Delta_{\alpha+1}$, hence, by (iv), $\Delta_{\alpha+1} = Cn(\emptyset)$. Since we presuppose that all conditionals with antecedents A such that $\vdash \neg A$ have been deleted from Γ in advance, it follows that $\{A_i \to B_i : \neg A_i \in \Delta_{\alpha+1}\} = \emptyset$, so by supposition $\{A_i \to B_i : \neg A_i \in \Delta_\alpha\} = \emptyset$.

(vi) From (v), we know that $\{A_i \to B_i : \neg A_i \in \Delta_\alpha\} - \{A_i \to B_i : \neg A_i \in \Delta_{\alpha+1}\} \neq \emptyset$, unless $\{A_i \to B_i : \neg A_i \in \Delta_\alpha\} = \emptyset$. That is, each step in the construction process reduces the number of conditionals to be taken into account by at least one. Let β be the smallest cardinal that is greater than the cardinal number of Γ. Since there is no bijective mapping between β and Γ, then $\{A_i \to B_i : \neg A_i \in \Delta_\beta\} = \emptyset$, so $\Delta_{\beta+1} = Cn(\emptyset)$. Hence the set of ordinals $\gamma \leq \beta+1$ such that $\Delta_\gamma = Cn(\emptyset)$ is non-empty. Let α be the smallest ordinal of that set. Then, by definition, $\top \in \alpha(\Gamma) = Cn_0(\emptyset)$. In particular, let Γ be finite and $|\Gamma| = n$; then $|\{A_i \to B_i : \neg A_i \in \Delta_0\}| = n$, and hence, since each step in the construction reduces the number of conditionals to be taken into account by at least one, $|\{A_i \to B_i : \neg A_i \in \Delta_n\}| = 0$. Thus $\Delta_{n+1} = Cn_0(\{A_i \to B_i : \neg A_i \in \Delta_n\}) = Cn_0(\emptyset)$, i.e., $\top \in \alpha(\Gamma)$ for some $\alpha \leq n+1$.

(vii) If $A \in Cn_0(\emptyset)$, then $A \in \alpha(\Gamma)$ if and only if α is the smallest ordinal such that $\Delta_\alpha = Cn_0(\emptyset)$, which exists, as shown above. If $A \notin Cn_0(\emptyset)$, then $A \notin \Delta_\alpha$ for those α such that $\Delta_\alpha = Cn_0(\emptyset)$. So the set of all ordinals β such that $A \notin \Delta_\beta$ is non-empty. Let γ be the smallest ordinal of that set. γ cannot be 0, for $\Delta_0 = L_0$; γ cannot be a limit ordinal for if $A \notin \Delta_\gamma$ for a limit ordinal γ then, by the construction of Δ_γ, there must be a $\gamma' < \gamma$ such that $A \notin \Delta_{\gamma'}$; so γ is a successor number. So $A \in \Delta_{\gamma-1} - \Delta_\gamma$, i.e., $A \in \gamma\text{-}1(\Gamma)$. To show uniqueness, suppose for reductio that $A \in \alpha(\Gamma)$ and $A \in \beta(\Gamma)$ with $\alpha < \beta$. From $A \in \alpha(\Gamma)$, it follows that $A \notin \Delta_{\alpha+1}$, hence, since $\Delta_\beta \subseteq \Delta_{\alpha+1}$, $A \notin \Delta_\beta$, contradicting $A \in \beta(\Gamma)$.

(viii) Let $A \in \Delta_\alpha$ and $A \leq_\Gamma B$. From $A \in \Delta_\alpha$, we get that $A \in \beta(\Gamma)$ for some $\beta \geq \alpha$, and from $A \leq_\Gamma B$ we then get that $B \in \gamma(\Gamma) = \Delta_\gamma - \Delta_{\gamma+1}$ for some $\gamma \geq \beta \geq \alpha$, hence, by (i), $B \in \Delta_\alpha$. So Δ_α is an L_0-cut with respect to $\leq_\Gamma$. To show that the Δ_α's are the only non-empty cuts with respect to $\leq_\Gamma$, suppose that $S \neq \emptyset$ is a cut with respect to $\leq_\Gamma$. Consider the set of ordinals β such that there is an $A \in S$ with $A \in \beta(\Gamma)$, and take the smallest ordinal α from this class. We show that $\Delta_\alpha = S$. Select some $A \in S$ with $A \in \alpha(\Gamma)$. Since S is a cut, $B \in L_0$ is in S iff $A \leq_\Gamma B$, i.e., iff $B \in \beta(\Gamma)$ for some $\beta \geq \alpha$, i.e., iff $B \in \bigcup\{\beta(\Gamma)\colon \beta \geq \alpha\} = \Delta_\alpha$, by (vii) and (ii).

(ix) From left to right: Let $A \leq_\Gamma B$ and $A \in \Delta_\alpha$. From the latter, we get that $A \in \beta(\Gamma)$ for some $\beta \geq \alpha$, then the former gives us that $B \in \gamma(\Gamma)$ for some $\gamma \geq \beta \geq \alpha$, hence $B \in \Delta_\gamma \subseteq \Delta_\alpha$, as desired. From right to left: Let $A \not\leq_\Gamma B$, i.e., by (vii), there are β and γ such that $A \in \beta(\Gamma)$, $B \in \gamma(\Gamma)$ and $\gamma < \beta$. But then, $A \in \Delta_\beta$ and $B \notin \Delta_{\gamma+1} \supseteq \Delta_\beta$, so A, but not B is in Δ_β, and we are done. □

Proof of Observation 11 (i) That E(Γ) satisfies Transitivity (E1) follows immediately from the transitivity of $\leq$ on the ordinals.

For Entailment (E2), suppose that $\emptyset \neq \Sigma \vdash A$ Consider the set of ordinals β such that there is a $B \in \Sigma$ with $\mathrm{rank}_\Gamma(B) = \beta$, take the smallest ordinal α from this class and select some $B \in \Sigma$ with $\mathrm{rank}_\Gamma(B) = \alpha$. Now since for all $C \in \Sigma$, $\mathrm{rank}_\Gamma(C) \geq \alpha$, they are in Δ_γ for some $\gamma \geq \alpha$, hence, by part (i) of Observation 10, each $C \in \Sigma$ is in Δ_α, so $\Sigma \subseteq \Delta_\alpha$. But Δ_α is closed under Cn_0 by definition. Hence $A \in \Delta_\alpha$, hence $A \in \gamma(\Gamma)$ for some $\gamma \geq \alpha$, hence $B \leq_\Gamma A$.

For Maximality (E3), suppose that $B \leq_\Gamma A$ for all B. That is, $\mathrm{rank}_\Gamma(B) \leq \mathrm{rank}_\Gamma(A)$ for all B, so in particular $\mathrm{rank}_\Gamma(\top) \leq \mathrm{rank}_\Gamma(A)$. Let $\alpha = \mathrm{rank}_\Gamma(\top)$. Then, by part (vi) of Observation 10, $\Delta_\alpha = Cn_0(\emptyset)$, and we have $\Delta_\beta = Cn_0(\emptyset)$ for $\beta > \alpha$, so, since $\mathrm{rank}_\Gamma(A) \geq \alpha$, $A \in Cn_0(\emptyset)$.

To show that E(Γ) is well-ordering, let Σ be a non-empty set of L_0-sentences. Consider the set $\{\alpha\colon \mathrm{rank}_\Gamma(A) = \alpha$ for some $A \in \Sigma\}$, select the smallest ordinal β from that set and a $B \in \Sigma$ such that $\mathrm{rank}_\Gamma(B) = \beta$. Since $\beta \leq \mathrm{rank}_\Gamma(A)$ for all $A \in \Sigma$, $B \leq_\Gamma A$ for all $A \in \Sigma$, so B is a smallest element in Σ, and we are done.

(ii) Suppose for reductio that there is an $A_i \mathbin{\Box\!\!\rightarrow} B_i \in \Gamma$ which $\leq_\Gamma$ does not satisfy. Then $A_i \to B_i \leq_\Gamma \neg A_i$, i.e., $\mathrm{rank}_\Gamma(A_i \to B_i) \leq \mathrm{rank}_\Gamma(\neg A_i)$. Let $\mathrm{rank}_\Gamma(\neg A_i) = \alpha$, i.e., $\neg A_i \in \Delta_\alpha - \Delta_{\alpha+1}$. As $\neg A_i \in \Delta_\alpha$, we get, by definition,

$A_i \to B_i \in \Delta_{\alpha+1} = Cn_0(\{A_j \to B_j \colon \neg A_j \in \Delta_\alpha\})$. So $\mathrm{rank}_\Gamma(A_i \to B_i) \geq \alpha+1 > \mathrm{rank}_\Gamma(\neg A_i)$, and we have a contradiction. □

Proof of Observation 12 Let $\mathrm{rank}_{E(\Gamma)}(A)=\alpha$, i.e., by definition $A \in L_0 - \overleftarrow{\alpha}(\leq_\Gamma)$ and $A \leq_\Gamma B$ for every $B \in L_0 - \overleftarrow{\alpha}(\leq_\Gamma)$. By part (ii) of Observation 10, this means that $A \in \Delta_\alpha$ and $A \leq_\Gamma B$ for every $B \in \Delta_\alpha$. Thus, by part (ix) of Observation 10, $A \in \Delta_\alpha$ and for every β and $B \in \Delta_\alpha$, if $A \in \Delta_\beta$ then $B \in \Delta_\beta$. But we know from parts (i) and (iv) of Observation 10 that $\alpha(\Gamma)$ is non-empty, i.e., that there is a $B \in \Delta_\alpha - \Delta_{\alpha+1}$. So $A \notin \Delta_{\alpha+1}$, so $A \in \alpha(\Gamma)$, so $\mathrm{rank}_\Gamma(A)=\alpha$. □

Proof of Observation 13 Let Γ be as indicated and $\leq$ an E-relation satisfying Γ. If $\leq$ is not well-ordering, then $E(\Gamma) \sqsubset_E \leq$ by definition, since $E(\Gamma)$ is well-ordering by Observation 11. So let $\leq$ be well-ordering. As $\sqsubseteq_E$ is antisymmetrical, it suffices to show that $E(\Gamma) \sqsubseteq_E \leq$. We show that for every L_0-sentence A, $\mathrm{rank}_{E(\Gamma)}(A) \leq \mathrm{rank}_\leq(A)$ by transfinite induction on $\mathrm{rank}_\leq(A)$.

Let, as induction hypothesis, $\mathrm{rank}_{E(\Gamma)}(A) \leq \mathrm{rank}_\leq(A)$ be established for all A with $\mathrm{rank}_\leq(A) < \alpha$. Now let $\mathrm{rank}_\leq(C)=\alpha$. We have to verify that $\mathrm{rank}_{E(\Gamma)}(C) \leq \alpha$, i.e., by Observation 12, $\mathrm{rank}_\Gamma(C) \leq \alpha$, i.e., $C \in \bigcup\{\beta(\Gamma) \colon \beta \leq \alpha\} = L_0 - \Delta_{\alpha+1}$, by part (ii) of Observation 10.

Suppose for reductio that $C \in \Delta_{\alpha+1}$, i.e., by construction, $C \in Cn_0\{A_i \to B_i \colon \neg A_i \in \Delta_\alpha\}$. Since $\leq$ is an E-relation, so by (E2), $A_i \to B_i \leq C$ for some i such that $\neg A_i \in \Delta_\alpha$. As $\leq$ satisfies Γ, we have $\neg A_i < A_i \to B_i$, so by (E1), $\neg A_i < C$. Hence, $\mathrm{rank}_\leq(\neg A_i) < \mathrm{rank}_\leq(C)=\alpha$. But $\neg A_i \in \Delta_\alpha$, so $\mathrm{rank}_{E(\Gamma)}(\neg A_i) = \mathrm{rank}_\Gamma(\neg A_i) \geq \alpha > \mathrm{rank}_\leq(\neg A_i)$, which contradicts the induction hypothesis. □

Proof of the Corollary First we recall that any set of sentences satisfied by a well-ordering E-relation $\leq$ is well-founded, so in particular $K(\leq)$ is well-founded and the construction of $E(K(\leq))$ yields a well-ordering E-relation, by Observation 11(i). In view of Observation 13 and the antisymmetry of $\sqsubseteq_E$, it suffices to show that $\leq \sqsubseteq_E E(K(\leq))$.

We show that $\mathrm{rank}_\leq(A) \leq \mathrm{rank}_{E(K(\leq))}(A)$ for every A by induction on $\mathrm{rank}_\leq(A)$. Let, as induction hypothesis, $\mathrm{rank}_\leq(A) \leq \mathrm{rank}_{E(K(\leq))}(A)$ be established for every A with $\mathrm{rank}_\leq(A) < \alpha$. Now let $\mathrm{rank}_\leq(A)=\alpha$. For $\alpha=0$, the claim is trivial. For $\alpha>0$, choose representatives B_β with $\mathrm{rank}_\leq(B_\beta)=\beta$ for all $\beta<\alpha$ (such B_β's exist!). Now clearly $B_\beta < A$ for all $\beta<\alpha$, hence $\leq$ satisfies $\neg A \vee \neg B_\beta \,\Box\!\!\to A$ for all $\beta<\alpha$. By Observation 11(ii), $E(K(\leq))$ satisfies $K(\leq)$, so $E(K(\leq))$ satisfies in particular $\neg A \vee \neg B_\beta \,\Box\!\!\to A$ for every $\beta<\alpha$, so $B_\beta \leq_{K(\leq)} A$ for every $\beta<\alpha$, so by induction hypothesis β=rank$_{\leq}$B$_{\beta}\leq$rank$_{E(K(\leq))}$B$_{\beta}<$rank$_{E(K(\leq))}$A for every $\beta<\alpha$, so rank$_{\leq}$(A)=$\alpha\leq$rank$_{E(K(\leq))}$A.

Proof of Observation 14 (i) Let A$\in$Cn$_0$(Γ)$\cap$L$_1$. As Cn$_0$ does not operate on the internal structure of conditionals, they may be regarded as atoms when applying Cn$_0$ to Γ. Note that since Γ is an L$_1$-set, these new atoms do not appear in any complex sentence. Therefore, if A is a conditional, i.e., if A is in L$_1$–L$_0$, it can only be a Cn$_0$-consequence of Γ if A$\in\Gamma$, and the claim reduces to Inclusion which will be proved as (iii). On the other hand, if A is in L$_0$, then, by the same argument, it must be a Cn$_0$-consequence of $\Gamma\cap$L$_0$. But as K$_0$($\leq$) is a Cn$_0$-theory for every E-relation $\leq$, every E-relation satisfying $\Gamma\supseteq\Gamma\cap$L$_0$ satisfies Cn$_0$($\Gamma\cap$L$_0$), so every E-relation for Γ satisfies A.

(ii) Let B$\in$L$_0\cap$Cn($\Gamma\cup\{$A$\}$). As B$\in$L$_0$, this means that B$\in$K$_0$(E($\Gamma\cup\{$A$\}$)) = Cn$_0$(L$_0$($\Gamma\cup\{$A$\}$)), by construction. But this is equivalent to A$\rightarrow$B$\in$ Cn$_0$(L$_0$(Γ)) = K$_0$(E(Γ)), i.e., A$\rightarrow$B$\in$Cn(Γ).

(iii)–(v) Immediate from the general results of Makinson (1989; 1990) for arbitrary preferential model structures.

(vi) As Makinson points out, it suffices to show that ($\mathcal{E}$,$\models$,$\sqsubset_E$) is "stoppered" in the sense that for all E-relations $\leq\in\mathcal{E}$ and all L$_1$-premise sets Γ, if $\leq\models\Gamma$ then there is a $\leq'$ such that either $\leq'\sqsubset_E\leq$ or $\leq'=\leq$, and $\leq'$ satisfies Γ $\sqsubset_E$-minimally. Let $\leq\models\Gamma$. In the first case, assume that there are only non-well-ordering E-relations for Γ. Then we are immediately done, since, by the definition of $\sqsubset_E$, non-well-ordering E-relations are incomparable with respect to $\sqsubset_E$, so $\leq'=\leq$ will do. So assume, as the second case, that there is a well-ordering E-relation for Γ. So Γ is well-founded. Hence E(Γ) is the smallest E-relation with respect to $\sqsubset_E$, so $\leq'$=E(Γ) will do.

(vii) Drawing again on Makinson's work, we only have to verify that $\sqsubset_E$ is transitive. But this follows trivially from the definition of $\sqsubset_E$. $\square$

Proof of Observation 15 By giving counterexamples.

(i) Let $\Gamma=\{\neg$A$\Box\!\!\rightarrow$B$\vee$D, $\neg$B$\Box\!\!\rightarrow$A$\vee$D$\}$; the $\leq$-translations are A$<$A$\vee$B$\vee$D and B$<$A$\vee$B$\vee$D. E($\Gamma\cup\{$A$\}$) has the E-base $\perp\prec$A$\prec$A$\vee$B$\vee$D$\prec\top$, E($\Gamma\cup\{$B$\}$) has the E-base $\perp\prec$B$\prec$A$\vee$B$\vee$D$\prec\top$; in both cases, it follows that A$\vee$B$<$A$\vee$B$\vee$D, so both $\Gamma\cup\{$A$\}$ and $\Gamma\cup\{$B$\}$ entail $\neg$(A$\vee$B)$\Box\!\!\rightarrow$D ($\equiv$C). But E($\Gamma\cup\{$A$\vee$B$\}$) has the E-base $\perp\prec$A$\vee$B$\prec\top$ which does not give A$\vee$B$<$A$\vee$B$\vee$D, so $\Gamma\cup\{$A$\vee$B$\}$ does not entail $\neg$(A$\vee$B)$\Box\!\!\rightarrow$D.

(ii) Let $\Gamma=\{\neg$C, A$\vee$C$\Box\!\!\rightarrow$D, $\neg$A$\vee$C$\Box\!\!\rightarrow$D$\}$; the $\leq$-translations are $\perp<\neg$C, $\neg$A&$\neg$C$<$($\neg$A&$\neg$C)$\vee$D and A&$\neg$C$<$(A&$\neg$C)$\vee$D. E($\Gamma\cup\{$A$\}$) has the E-base $\perp\prec$A$\simeq\neg$C$\simeq$D$\prec$(A&$\neg$C)$\vee$D$\prec\top$, E($\Gamma\cup\{\neg$A$\}$) has the E-base

$\bot\prec\neg A\simeq\neg C\simeq D\prec(\neg A\&\neg C)\vee D\prec\top$; in both cases, it follows that $\neg C<\neg C\vee D$, so both $\Gamma\cup\{A\}$ and $\Gamma\cup\{\neg A\}$ entail $C\Box\!\!\rightarrow D$ ($\equiv B$). But $E(\Gamma)$ has the E-base $\bot\prec\neg C\simeq D\prec\top$ which does not give $\neg C<\neg C\vee D$, so Γ does not entail $C\Box\!\!\rightarrow D$.

(iii) Let $\Gamma=\{C,\ D,\ \neg D\Box\!\!\rightarrow C,\ \neg A\vee\neg C\Box\!\!\rightarrow C,\ A\vee\neg C\Box\!\!\rightarrow C\}$; the $\leq$-translations are $\bot<C$, $\bot<D$, $D<C\vee D$, $A<C$ and $\neg A<C$. $E(\Gamma)$ has the E-base $\bot\prec C\simeq D\prec C\vee D\prec\top$, so Γ entails $\neg C\Box\!\!\rightarrow D$ ($\equiv B$). But $E(\Gamma\cup\{A\})$ has the E-base $\bot\prec A\simeq D\prec C\simeq C\vee D\prec\top$, and $E(\Gamma\cup\{\neg A\})$ has the E-base $\bot\prec\neg A\simeq D\prec C\simeq C\vee D\prec\top$, so neither gives $C<C\vee D$, so neither $\Gamma\cup\{A\}$ nor $\Gamma\cup\{\neg A\}$ entails $\neg C\Box\!\!\rightarrow D$.

(iv) Let $\Gamma=\{A,\neg A\vee\neg C\Box\!\!\rightarrow C\}$. Γ is consistent and has the E-base $\bot\prec A\prec C\prec\top$. $Cn(\{A\})$ is the set of all sentences satisfied by $E(\{A\})$ which has the E-base $\bot\prec A\prec\top$. Now $\neg A\vee\neg C\Box\!\!\rightarrow A\in Cn(\{A\})$. So both $\neg A\vee\neg C\Box\!\!\rightarrow A$ and $\neg A\vee\neg C\Box\!\!\rightarrow C$ ($\equiv B$) are in $Cn(\{A\})\cup\{\neg A\vee\neg C\Box\!\!\rightarrow C\}$. But there is no E-relation satisfying these two sentences, for the first one translates to $A<C$ while the second one translates to $C<A$ which contradicts the definition of $<$ from $\leq$. $\Box$

Proof of Observation 16 We restrict our attention to the principal cases with antecedents the negations of which are not in $Cn_0(\emptyset)$; the limiting cases are all trivial. In view of Observation 4, we can replace $\models$ by $\models_1$.

(i) $\models_1 A$ for $A\in L_0\cap Cn_0(\emptyset)$: $\bot<\top\doteq A$ for such an A and every E-relation $\leq$, by (E3).

(ii) $\{A\Box\!\!\rightarrow B,\ A\Box\!\!\rightarrow C\}\models_1 A\Box\!\!\rightarrow(B\&C)$: Let $\leq$ satisfy the premises, i.e., $\neg A<\neg A\vee B$ and $\neg A<\neg A\vee C$. By (E2) then, $\neg A\vee B\leq A\vee(B\&C)$ or $\neg A\vee C\leq A\vee(B\&C)$, so, by (E1) and (E2), $\neg A<\neg A\vee(B\&C)$, i.e., $\leq\models_1 A\Box\!\!\rightarrow(B\&C)$.

(iii) $\models_1 A\Box\!\!\rightarrow\top$: like (i), $\neg A<\neg A\vee\top$ is immediate, provided that $\neg A\notin Cn_0(\emptyset)$.

(iv) $\models_1 A\Box\!\!\rightarrow A$: like (i), $\neg A<\neg A\vee A$ is immediate, provided that $\neg A\notin Cn_0(\emptyset)$.

(v) $\{A\Box\!\!\rightarrow B)\}\models_1 A\rightarrow B$: Let $\leq$ satisfy the premise, i.e., $\neg A<\neg A\vee B$. Thus $\bot\leq\neg A<\neg A\vee B$, so $\bot<A\rightarrow B$, i.e., $\leq\models A\rightarrow B$.

(vi) $\{A\&B\}\models_1 A\Box\!\!\rightarrow B$: Let $\leq$ satisfy the premise, i.e., $\bot<A\&B$. Then, by (E2), $\neg A\leq\bot<A\&B\leq\neg A\vee B$, so $\leq\models A\Box\!\!\rightarrow B$.

(vii) $\{A\Box\!\!\rightarrow\neg A\}\models_1 B\Box\!\!\rightarrow\neg A$: Immediate, since there is no $\leq$ satisfying $A\Box\!\!\rightarrow\neg A$ which would mean $\neg A<\neg A\vee\neg A$, a contradiction with (E2).

(viii) $\{A\Box\!\!\rightarrow B,\ B\Box\!\!\rightarrow A,\ A\Box\!\!\rightarrow C\}\models_1 B\Box\!\!\rightarrow C$: Let $\leq$ satisfy the premises, i.e., $\neg A<\neg A\vee B$, $\neg B<\neg B\vee A$, and $\neg A<\neg A\vee C$. The first term implies, with the help of (E2), that $\neg B\leq\neg A$. Also by (E2), either $\neg B\vee A\leq\neg B\vee C$ or

$\neg A\vee C\leq\neg B\vee C$. In the former case, we get $\neg B<\neg B\vee A\leq\neg B\vee C$, in the latter case we get $\neg B\leq\neg A<\neg A\vee C\leq\neg B\vee C$, so in any case $\leq\models B\Box\!\!\rightarrow C$.

(ix) $\{A\Box\!\!\rightarrow C, B\Box\!\!\rightarrow C\}\models_1 A\vee B\Box\!\!\rightarrow C$: Let $\leq$ satisfy the premises, i.e., $\neg A<\neg A\vee C$ and $\neg B<\neg B\vee C$. By (E2), either $\neg A\vee C\leq(\neg A\&\neg B)\vee C$ or $\neg B\vee C\leq(\neg A\&\neg B)\vee C$. In the former case, $\neg A\&\neg B\leq\neg A<\neg A\vee C\leq(\neg A\&\neg B)\vee C$, in the latter case similarly $\neg A\&\neg B\leq\neg B<\neg B\vee C\leq(\neg A\&\neg B)\vee C$, so in any case $\leq\models(A\vee B)\Box\!\!\rightarrow C$.

(x) $\{A\Box\!\!\rightarrow C, \neg(A\Box\!\!\rightarrow\neg B)\}\models_1 A\&B\Box\!\!\rightarrow C$: Let $\leq$ satisfy the premises, so $\neg A<\neg A\vee C$ and, by the consistency condition for L_2, $\neg A\vee\neg B\leq\neg A$. Hence, by (E1) and (E2), $\neg A\vee\neg B\leq\neg A<\neg A\vee C\leq(\neg A\vee\neg B)\vee C$, so $\leq\models(A\&B)\Box\!\!\rightarrow C$.

(xi) If $C\in Cn_0(B)$ then $A\Box\!\!\rightarrow B\models_1 A\Box\!\!\rightarrow C$: Let $C\in Cn_0(B)$ and let $\leq$ satisfy the premise, i.e., $\neg A<\neg A\vee B$. Since $C\in Cn_0(B)$, also $\neg A\vee C\in Cn_0(\neg A\vee B)$, so, by (E2), $\neg A\vee B\leq\neg A\vee C$, hence $\neg A<\neg A\vee C$, i.e., $\leq\models A\Box\!\!\rightarrow C$. $\Box$

Proof of Observation 17 (i) We show the positive claims concerning $\models$; the negative claims concerning $\not\models$ follow from (ii). The E-relations $E(\Gamma)$ generated by the respective premise sets of (SA), (Tr) and (Cp) have the following E-bases:

$\bot\prec\neg A\vee B\prec\top$,

$\bot\prec\neg A\vee B\simeq\neg B\vee C\prec\top$,

$\bot\prec\neg A\vee B\prec\top$, respectively.

Clearly, for these E-relations it holds that $\neg A\vee\neg C<\neg A\vee\neg C\vee B$, $\neg A<\neg A\vee C$, and $B<\neg A\vee B$, respectively. So they satisfy the desired conclusions.

(ii) The E-relations $E(\Gamma)$ generated by the augmented premise sets have the following E-bases:

$\bot\prec\neg A\vee B\simeq\neg A\vee\neg C\prec\top$,

$\bot\prec\neg A\simeq\neg B\vee C\prec\neg A\vee B\prec\top$,

$\bot\prec B\prec\top$, respectively.

It is easy to see that for these E-relations it holds that $\neg A\vee\neg C\vee B\leq\neg A\vee\neg C$, $\neg A\vee C\leq\neg A$, and $\neg A\vee B\leq B$, respectively. So they do not satisfy the desired conclusions.

(iii) Let in the following $\leq$ satisfy the premise(s) but not the conclusion of our inference patterns.

In the case of (SA), we have $\neg A<\neg A\vee B\leq\neg A\vee\neg C\vee B\leq\neg A\vee\neg C$, so $\leq$ satisfies $A\Box\!\!\rightarrow\neg C$.

In the case of (Tr), we have, first, $\neg A<\neg A\vee B$, secondly, $\neg B<\neg B\vee C$, and thirdly, $\neg A\vee C\leq\neg A$. From (E2), we know that either $\neg A\vee B\leq\neg A\vee C$ or $\neg B\vee C\leq\neg A\vee C$. But the former cannot be, since it would imply, with the help

of the first and third condition, ¬A<¬A. So ¬B∨C≤¬A∨C. But by the second and the third condition and by (E2) this gives us the chain ¬B<¬B∨C≤¬A∨C≤¬A≤¬B∨¬A, so ≤ satisfies B□→¬A.
In the case of (Cp), we have ⊥≤¬A<¬A∨B≤B, so ≤ satisfies B. □

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