

# Degrees all the way down: Beliefs, non-beliefs and disbeliefs

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## 1 INTRODUCTION

A Cartesian skeptic must not accept anything but what is ideally clear and distinct in her mind. She has only few beliefs, but all her beliefs have maximal certainty.<sup>1</sup> Some philosophers recommend that a responsible believer should believe only what is beyond any doubt, namely logical and analytical truths.<sup>2</sup> The common core of such proposals is that maximal certainty is held to be necessary and sufficient for rational belief. Consequently, the believer's beliefs are all on an equal footing, no degrees of belief are needed, the set of beliefs is 'flat'.

Other philosophers felt that this picture is inadequate. As a matter of fact, we are not as reluctant to take on beliefs as Descartes admonished us to be. We believe quite a lot of things, and we are aware that there are differences in the quality of these beliefs. There is no denying that we are fallible, and if we are forced to give up some of the beliefs we have formed (Descartes would say, precipitately), we can adapt our beliefs in accordance with their varying credentials. Beliefs can be thought of as being equipped with labels specifying their 'certainty', or, to use a different terminology, their 'entrenchment' in a person's belief state.

Ever since Hintikka (1962), philosophers and logicians have been fond of thinking of belief as a form of necessity ('doxastic necessity'). If one wants to acknowledge distinctions in degrees of belief, one has to introduce a notion of *comparative necessity*. Saying that *A* is more firmly believed than *B* is to say that *A* is more necessary than *B*. Degrees of belief are grades of modality.

Having said that, there are questions on two sides. First, it is of course natural to think of a dual to necessity, a doxastic possibility operator. Second, we seem to miss rankings of non-beliefs. Even if a proposition is not believed, it may be more or less close to being believed. We shall distinguish *non-beliefs* in

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<sup>1</sup>Actually, not so few and not so certain beliefs if she runs through the six *Meditations* to their very end.

<sup>2</sup>'Relations of ideas', in Hume's favourite terms. Isaac Levi calls such an epistemology 'Parmenidean'.

a narrow sense from *disbeliefs* (see Quine and Ullian 1978, p. 12). A sentence  $A$  is disbelieved by a person if she believes the negation of  $A$ , and the person is in a state of non-belief with respect to  $A$  if she is agnostic about it, i.e., believes neither  $A$  nor the negation of  $A$ .<sup>3</sup> In a rather straightforward manner, the notion of doxastic possibility can be applied to non-beliefs:  $A$  is a non-belief if and only if the agent considers both  $A$  and the negation of  $A$  possible. Perhaps surprisingly, we shall see that it has also been quite common to apply the notion of doxastic possibility to disbeliefs.

To believe that  $A$  is true means that  $A$  is true in *all* the worlds that the person regards as (doxastically) possible, and truth in all accessible possible worlds has long been viewed as explicating the notion of necessity. As a consequence of the classical notion of a world, it follows that the reasoner believes  $A \wedge B$  if and only if she believes  $A$  and believes  $B$ . Necessity distributes over conjunction. Possibility is dual to necessity, and thus it distributes over disjunction. The reasoner believes that  $A \vee B$  is possible if and only if she believes that  $A$  is possible or believes that  $B$  is possible. We shall see that in a framework that acknowledges degrees of belief, comparative necessity comes with a special condition for conjunctions, and comparative possibility with a special condition for disjunctions.

It is not quite clear whether the term ‘degree of belief’ should in the first instance be applied to beliefs or non-beliefs. Advocates of subjective probability theory, identifying degrees of belief with *probability* values, seem to apply it primarily to non-beliefs. If your degree of belief of  $A$  is represented by probability 0.5, for example, you certainly don’t believe that  $A$ . In contrast, I start this paper by assuming that degrees of belief should in the first instance be applied to beliefs.<sup>4</sup> The paradigm case of a degree of belief gives expression to the *firmness* of belief.

It is the main aim of the present paper, however, to extend the degrees of *belief* into a single linear ordering that also applies in a non-trivial way to both disbeliefs and non-beliefs. It will turn out that the ways of constructing the two extensions are surprisingly uniform, viz., by relational intersection, even if the results differ in structure. A constraint to be met is that beliefs should be ranked higher than disbeliefs, and that non-beliefs in the narrow sense should find a place between beliefs and disbeliefs.<sup>5</sup>

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<sup>3</sup>Thus ‘ $A$  is not a belief’ is equivalent to saying that ‘ $A$  is either a non-belief or a disbelief’. To avoid confusion, I shall sometimes speak of ‘non-beliefs in the narrow sense’ rather than just ‘non-beliefs’.

<sup>4</sup>Here I side with Levi (1984, p. 216): “In presystematic discourse, to say that  $X$  believes that  $h$  to a positive degree is to assert that  $X$  believes that  $h$ .”

<sup>5</sup>The notion of *partial belief* which is often used in probabilistic frameworks will not be addressed in the present paper. We shall have no measure function that could express the relative size of a proposition in a reasoner’s space of doxastic possibilities. Degrees are not proportions.

The term ‘degrees of belief’ is usually taken to mean that some numbers, for instance probabilities, are assigned to the beliefs in question. I shall not make this assumption in the present paper. It is rather hard to justify the assignment of a precise numerical value. Often it is easier (if less informative) to determine the certainty of a belief not absolutely, but only in comparison with other beliefs. I shall be content in this paper with the comparative notion of ‘degree’, as expressed in the phrase ‘I believe  $A$  to a greater degree than  $B$ ’.

Whenever I will make use of numerals in the following, they are not meant to stand for genuine numbers representing metrical relations among beliefs. They just serve as convenient indicators of positions in a total preordering. The operations of addition, subtraction or multiplication wouldn’t make any sense if applied to the numerals I will be using.<sup>6</sup>

Non-probabilistic degrees of belief have sometimes been advocated because in contrast to ordinary probability, they allow us to model ‘plain belief’ (Spohn 1990; 1991, p. 168) or ‘total ignorance’ (Dubois, Prade and Smets 1996). At the end of this paper we will return to the question of the meaning of ‘belief’. Having extended a qualitative notion of degree to both non-beliefs and disbeliefs, I will argue that the notion of belief (as well as the correlative notions of disbelief and non-belief) is as elusive here as it is in the probabilistic context.

## 2 DEGREES OF BELIEFS

**2.1 Entrenchment relations.** We begin the presentation with a way of distinguishing *beliefs* according to their degrees of firmness, certainty or endorsement. A measure of the firmness of belief can be seen in their invulnerability, that is, the resistance they offer against being given up. Such measures are provided in the work on *entrenchment relations* by Peter Gärdenfors and David Makinson (Gärdenfors 1988, Gärdenfors and Makinson 1988) and on the *degrees of incorrigibility* by Isaac Levi (see Levi 1996, p. 264; 2004, pp. 191–199). Dubois and Prade (1991) have rightly pointed out that entrenchment is a notion of comparative necessity and have related it to their own account of possibility theory.<sup>7</sup>

In the following, we propose to read  $A \leq_e B$  as “ $B$  is at least as firmly believed as  $A$ ” or “ $B$  is at least as entrenched among the reasoner’s beliefs as  $A$ ”. Here are the first three Gärdenfors-Makinson axioms for entrenchment relations<sup>8</sup>

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<sup>6</sup>An exception seems come up in Section 4.2, but as remarked there, the plus sign used in the construction of ‘belief functions’ from ‘entrenchment’ and ‘plausibility functions’ is not necessary, but serves just a convenient means for encoding an operation that could just as well be presented in purely relational terms.

<sup>7</sup>Possibility theory as developed in Toulouse has produced a large number of very important contributions to the topics covered in this paper. See for instance Dubois (1986), Dubois and Prade (1988a), Dubois, Prade and Smets (1996) and Dubois, Fargier and Prade (2004).

<sup>8</sup>This set of axioms is exactly the one used by Gärdenfors and Makinson (1994, p. 210) for *expectation* orderings. I will offer an account of the difference between beliefs and expectations later in this paper.

- (E1) If  $A \leq_e B$  and  $B \leq_e C$  then  $A \leq_e C$  *Transitivity*
- (E2) If  $A \vdash B$  then  $A \leq_e B$  *Dominance*
- (E3)  $A \leq_e A \wedge B$  or  $B \leq_e A \wedge B$  *Conjunctiveness*

We work in a purely propositional language, containing the truth and falsity constants  $\top$  and  $\perp$ . For the sake of simplicity, we assume that the background logic is classical (or of some similar Tarskian kind). Thus in (E2),  $\vdash$  may be thought of as denoting the consequence relation of classical logic. Condition (E3) establishes a kind of functionality of  $\leq_e$  with respect to conjunction, since the converse inequalities  $A \wedge B \leq_e A$  and  $A \wedge B \leq_e B$  already follow from (E2). The firmness of belief of a conjunction equals that of the weaker conjunct.

It is easy to derive from these axioms that entrenchment relations are total.

$$A \leq_e B \text{ or } B \leq_e A.$$

Thus any two sentences can be compared in terms of entrenchment. The very talk of ‘degrees of belief’ seems to presuppose this.<sup>9</sup>

By (E2), logically equivalent sentences are equally entrenched, they have the same ‘degree of belief’. The contradiction  $\perp$  is minimally and the tautology  $\top$  is maximally entrenched.

The entrenchment of a proposition is determined by the least incisive way of making that proposition false. This explains (E3), for instance. A least incisive way of making  $A \wedge B$  false is either a least incisive way of making  $A$  false or a least incisive way of making  $B$  false (or both). For reasons explained in Section 4, there can be no corresponding condition for disjunctions, a disjunction  $A \vee B$  can indeed be strictly more entrenched than either of its disjuncts. There is no condition for negation, but it is easy to deduce from (E1)–(E3) that at least one of  $A$  and  $\neg A$  is minimal with respect to  $\leq_e$ .

We treat as optional two conditions of Gärdenfors and Makinson (1988) relating to the maximum and minimum of entrenchments.

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<sup>9</sup>Many people find this too strong. We often appear to have beliefs that we are unable to rank in terms of firmness or certainty. For instance, I am more certain that I can jump 3 meters than that I can jump 3.5 meters. But how does this relate to the question whether Henry V is the father of Henry VI? It seems that I am neither more certain that Henry V is the father of Henry VI than that I can jump 3.5 meters, nor am I more certain that I can jump 3 meters than that Henry V is the father of Henry VI. I simply feel that I cannot compare the historical belief with my beliefs about my ability to leap over a certain length. I agree that full comparability of entrenchments is a very strong and sometimes unrealistic requirement. Various concepts of entrenchment without comparability are studied by Lindström and Rabinowicz (1991) and Rott (1992, 2000, 2003). The choice-theoretically motivated condition for the most ‘basic’ entrenchment (Rott 2001, p. 233, Rott 2003), viz.,  $A \wedge B \leq_e C$  if and only if both  $A \leq_e B \wedge C$  or  $B \leq_e A \wedge C$ , has turned out to be dual to Halpern’s (1997) condition of ‘qualitativeness’. Still, for the purposes of this paper, I will assume that all beliefs are comparable in terms of their firmness.

(E4)  $\perp <_e A$  iff  $A$  is believed *Minimality*

(E5)  $\top \leq_e A$  only if  $\vdash A$  *Maximality*

(E4) can be considered to be an explication of the notion of *belief*: A sentence is entrenched in a person's belief state to any non-minimal degree if and only if it is believed to a degree that exceeds that of the non-beliefs.<sup>10</sup> Put equivalently, with the help of (E1)–(E3), that  $A$  is believed means that  $\neg A <_e A$ . Beliefs are more entrenched than their negations. Non-beliefs (in the wide sense including disbeliefs), on the other hand, are only minimally 'entrenched', i.e., as entrenched as  $\perp$ . Only beliefs are really ranked by the degrees-of-*belief* relation  $\leq_e$  which offers nothing to distinguish between non-beliefs.

The 'Parmenidean' condition (E5) says that only tautologies are maximally entrenched. Setting technical advantages aside, there is little to recommend this condition. Let us call maximally entrenched sentences *a priori*. I do not see that we should dogmatically deny logically contingent sentences the status of apriority. But (E5) implies the much weaker

(E5') Not  $\top \leq_e \perp$  *Non-triviality*

which is a very reasonable condition. If  $\top \leq_e \perp$ , then all sentences of the language have the same entrenchment which trivializes the notion of a degree, and the reasoner would be at a loss whether to believe everything or nothing. Next to this trivial relation are entrenchments having exactly two layers, one including  $\top$  and one including  $\perp$ , the former containing all beliefs (which are all 'a priori') and the latter containing all non-beliefs (in the wide sense). Although not trivial, such *two-layered* relations do not represent substantive degrees of belief either, but express nothing more than a categorical yes-no notion of belief.<sup>11</sup>

**2.2 Entrenchment ranking functions.** There are situations in which the requirements on the representation of firmness have to be tightened, situations in which one does not only *compare* beliefs but in which one wants to distinguish *distances* in strength or degree of belief. Rather than just saying that  $A$  is less firmly believed or less entrenched than  $B$ , one wants to express how much less firmly  $A$  is believed than  $B$ . To this end, one can map beliefs onto a scale, i.e., a totally ordered set of numbers, like the natural numbers (Spohn) or the closed real interval from 0 to 1 (Dubois and Prade). Rather than just saying that  $A <_e B$ , one might say, for instance, that the degree of entrenchment (as a degree of belief) of  $A$  is 3, say, while the degree of entrenchment of  $B$  is 8. And this of course is meant to express more of a difference than the degree 4 for  $A$  and a degree 6 for  $B$ , even though the purely relational term is  $A <_e B$  both times.

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<sup>10</sup>We may safely neglect here Gärdenfors and Makinson's restriction of (E4) to consistent belief sets.

<sup>11</sup>According to my idealized description at the beginning of this paper, every skeptic has a two-layered entrenchment relation. But the converse is of course not true; not every person with a two-layered degree structure has beliefs that are absolutely certain.

We follow Spohn in favouring the discrete structure of the integers.

An *entrenchment ranking function*, or simply *entrenchment function*,  $\varepsilon$  assigns to each sentence a non-negative integer such that

- (Ei) Sentences equivalent under  $\vdash$  get the same  $\varepsilon$ -value. *Intensionality*
- (Eii)  $\varepsilon(\perp) = 0$  *Bottom*
- (Eiii)  $\varepsilon(A \wedge B) = \min\{\varepsilon(A), \varepsilon(B)\}$  *Conjunctiveness*

Entrenchment functions express quantitative degrees of beliefs. Condition (Ei) needs no comment. Condition (Eii) is not really necessary and listed here just for the sake of convenience and continuity with the literature. Condition (Eiii) is the most characteristic feature of entrenchment ranking functions, essentially expressing what the axioms (E2) and (E3) above express in qualitative terms.

It follows from these conditions that for any  $A$ , either  $\varepsilon(A) = 0$  or  $\varepsilon(\neg A) = 0$ , corresponding to the fact that either  $A \leq \perp$  or  $\neg A \leq \perp$  for entrenchment relations.

Spohn's idea is that a positive entrenchment value  $\varepsilon(A) > 0$  means that  $A$  is believed. A maximum entrenchment value  $\varepsilon(A) \geq \varepsilon(\top)$  means that  $A$  is a priori for an agent with the doxastic state represented by  $\varepsilon$ . One may stipulate again that only tautologies are a priori, or, more cautiously, add at least the non-triviality condition  $\varepsilon(\perp) < \varepsilon(\top)$ . We take all these conditions as optional.

Variants of entrenchment functions were introduced with Shackle's (1949, 1961) 'degrees of belief', Levi's (1967) 'degrees of confidence of acceptance', Cohen's (1977) 'Baconian probability', Rescher's (1964, 1976) 'conjunction-closed modal categories' and 'plausibility indexings', Shafer's (1976) 'consonant belief functions', Dubois and Prade's (1988a, 1988b, 1991) 'necessity measures', Gärdenfors and Makinson's (1994) 'belief valuations' and Williams's (1995) 'partial entrenchment rankings'.

**2.3 Entrenchment functions and relations.** Let us call a reflexive and transitive relation  $\leq$  *finite* if the symmetric relation  $\simeq = \leq \cap \leq^{-1}$  partitions the field of the relation into finitely many equivalence classes.

**Observation 1** *Take an entrenchment function  $\varepsilon$ . Then its relational projection defined by*

$$A \leq_e B \text{ iff } \varepsilon(A) \leq \varepsilon(B)$$

*is an entrenchment relation. Conversely, for every finite entrenchment relation  $\leq_e$  there is an entrenchment function  $\varepsilon$  such that  $\leq_e$  is the relational projection of  $\varepsilon$ .*

I think it is fair to say that the first part of this observation it is folklore in belief revision theory. The second part is equally simple. Just take the equivalence classes with respect to  $\simeq_e$  and number them, beginning with 0, "from the

bottom to the top” according to the ordering induced by  $\leq_e$ . The structure of the entrenchment relation guarantees that the function generated satisfies (Ei)–(Eiii).

Notice that if we start from  $\varepsilon$ , take its projection  $\leq_e$  and afterwards apply the construction just mentioned to  $\leq_e$ , then in general we will *not* get back  $\varepsilon$  again. All the “gaps” in  $\varepsilon$  will be closed in the new entrenchment function.

Now we have quite a fine-grained and satisfactory notion of degrees of *belief*. The problem is, however, that the propositions that are not believed are all on the same level. They are all as entrenched as the contradiction  $\perp$ . Intuitively this seems just wrong. Believers do make distinctions between non-beliefs just as elaborately as between beliefs. We begin to address this modelling task by refining the degrees of disbelieved sentences. This will be our first step in fanning out the ‘lowest’ degree of belief. The second step will then continue by fanning out the newly formed, still large ‘middle’ layer of non-beliefs in the narrow sense.

### 3 DEGREES OF DISBELIEFS

Do we have any means to rank the large class of sentences at the bottom of entrenchment? The sentences that are not believed fall into two classes. On the non-beliefs in the narrow sense, the reasoner does not take any firm stand. The disbeliefs, on the other hand, are sentences that the reasoner believes to be false. Among the latter, we can distinguish various degrees of plausibility. The key idea, it turns out, is to tie the notion of the plausibility of a disbelief to the entrenchment of its negation. Degrees of disbelief are in a sense dual to degrees of belief.

**3.1 Plausibility relations.** Let us first look at the binary relation that compares degrees of disbeliefs. We propose to read  $A \leq_p B$  as “ $A$  is at most as plausible as  $B$ ” or “ $B$  is at least as close to the reasoner’s beliefs as  $A$ ”. Plausibility has the same direction as entrenchment. The “better” the doxastic status of a proposition (either in terms of entrenchment or in terms of plausibility), the higher it is in the relevant ordering.

- |      |  |                        |
|------|--|------------------------|
| (P1) | If $A \leq_p B$ and $B \leq_p C$ then $A \leq_p C$ | <i>Transitivity</i>    |
| (P2) | If $A \vdash B$ then $A \leq_p B$                  | <i>Dominance</i>       |
| (P3) | $A \vee B \leq_p A$ or $A \vee B \leq_p B$         | <i>Disjunctiveness</i> |

(P1) and (P2) are identical with (E1) and (E2). (P3) is dual to (E3), with disjunction playing the role of conjunction. Conditions (P2) and (P3) together establish a kind of functionality of  $\leq_p$  with respect to disjunction. The degree of plausibility of a disjunction equals that of the more plausible disjunct. Like in the case of entrenchment, (P1)–(P3) taken together immediately entail that plausibility relations are total.

Again like in the case of entrenchment, we treat as optional two conditions concerning the maximum and minimum of plausibility.

(P4)  $\top \leq_p A$  iff  $\neg A$  is not believed *Maximality*

(P5)  $A \leq_p \perp$  only if  $\vdash \neg A$  *Minimality*

(P4) can be considered to be an explication of the notion of *believing-possible*: A sentence is maximally plausible in a person's belief state if and only if it is not excluded as impossible by the person's belief state. Put equivalently, with the help of (P1)–(P3), that  $A$  is believed means that  $\neg A <_p \top$ , or equivalently, that  $\neg A <_p A$ . All the sentences that are believed possible, i.e., not disbelieved, are maximally plausible, and in this respect there is nothing to distinguish between them. Distinctions in plausibility are only made between sentences that are disbelieved.

(P5) says that only contradictions are minimally plausible, a condition that we do not want to endorse as universally valid. (P5) implies the much weaker

(P5') Not  $\top \leq_p \perp$  *Non-triviality*

which is a very reasonable condition.

The plausibility of a proposition is determined by the most plausible way of making that proposition true. This explains (P3), for instance. A most plausible way of making  $A \vee B$  true is either a most plausible way of making  $A$  true or a most plausible way of making  $B$  true (or both). For reasons explained in Section 4, there can be no corresponding condition for conjunctions, a conjunction  $A \wedge B$  can indeed be strictly less plausible than either its conjuncts. There is no condition for negation, but it is easy to deduce from (P1)–(P3) that at least one of  $A$  and  $\neg A$  is maximal with respect to  $\leq_p$ .

A disbelief  $A$  is *less plausible* than another disbelief  $B$  if and only if the negation of the former has a higher degree of belief, i.e., is *more entrenched*, than the negation of the latter.

**Observation 2** *Take an entrenchment relation  $\leq_e$ . Then its dual defined by*

$$A \leq_p B \text{ iff } \neg B \leq_e \neg A$$

*is a plausibility relation. And vice versa.*

This duality makes clear that while entrenchment relations are comparative necessity relations, plausibility relations are comparative possibility relations. The conjunctive condition for entrenchments is changed into a disjunctive condition for plausibilities by the occurrence of the negations in Observation 2.

**3.2 Plausibility ranking functions.** We now have a look at numerical ranks of disbeliefs. Although there is a line of predecessors going back to Shackle (1949), I take Spohn (1988) as the seminal paper for this model. The direction



of Spohnian  $\kappa$ -functions, however, is reversed in relation to plausibility relations. For Spohn, lower ranks (that can be thought of as closer to the person's current beliefs) denote higher plausibility, higher ranks are 'farther off' or 'more far-fetched'. For reasons that will become clear later, we want more plausible (or 'more possible') sentences to get higher ranks. Thus we introduce *plausibility ranking functions*, or simply *plausibility functions*, that are the negative mirror images of Spohnian  $\kappa$ -functions (which could be called *implausibility functions*).

A plausibility function  $\pi$  assigns to each sentence a non-positive integer such that:

- (Pi) Sentences equivalent under  $\vdash$  get the same  $\pi$ -value. *Intensionality*
- (Pii)  $\pi(\top) = 0$  *Top*
- (Piii)  $\pi(A \vee B) = \max\{\pi(A), \pi(B)\}$  *Disjunctiveness*

Plausibility functions express quantitative degrees of disbelief. Condition (Pii) is not really necessary. Condition (Piii) is the most characteristic feature of plausibility ranking functions, expressing what the axioms (P2) and (P3) above express in qualitative terms.

It follows from these conditions that either  $\pi(A) = 0$  or  $\pi(\neg A) = 0$ . A negative plausibility value  $\pi(A) < 0$  means that  $A$  is disbelieved. A minimal plausibility value  $\pi(A) \leq \pi(\perp)$  means that  $\neg A$  is a priori.

Plausibility functions are entirely dual to entrenchment functions.

**Observation 3** *If  $\varepsilon$  is an entrenchment function and  $\pi$  is defined by*

$$\pi(A) = -\varepsilon(\neg A)$$

*then  $\pi$  is a plausibility function. And vice versa.*

If  $\pi$  and  $\varepsilon$  are related as in Observation 3, then either  $\pi(A) = 0$  or  $\varepsilon(A) = 0$ , i.e., either  $A$  or  $\neg A$  is doxastically possible.

**3.3 Plausibility functions and plausibility relations.** The following observation is entirely dual to Observation 1.

**Observation 4** *Take a plausibility function  $\pi$ . Then its relational projection defined by*

$$A \leq_p B \text{ iff } \pi(A) \leq \pi(B)$$

*is a plausibility relation. Conversely, for every finite plausibility relation  $\leq_p$  there is a plausibility function  $\pi$  such that  $\leq_p$  is the relational projection of  $\pi$ .*

Again, I think it is fair to say that this is folklore in belief revision theory.

## 4 COMBINING DEGREES OF BELIEF AND DEGREES OF DISBELIEF

The axiom sets (E1)–(E3) and (P1)–(P3) are very similar. It is tempting to ‘combine’ degrees of belief and disbelief by just collecting their axioms, and dropping the subscripts ‘ $e$ ’ and ‘ $p$ ’ attached to ‘ $\leq$ ’. Combining the minimum clause for conjunctions with the maximum clause for disjunctions would make degrees of (dis-)beliefs more “truth-functional.” However, we can show that given the background in which  $\vdash$  is the classical consequence relation, the combining of the requirements for entrenchments and plausibilities results in a trivialization.<sup>12</sup>

**Observation 5** *If a relation  $\leq$  satisfies (E1)–(E3) and (P3), then it is at most two-layered.*

We can see this as follows. Suppose that  $A < \top$  and  $B < \top$  are two arbitrarily chosen sentences of non-maximal entrenchment. Then  $A \vee B < \top$ , by (P3) and (E1). Since  $\top$  is equivalent with  $(A \vee B) \vee (\neg A \wedge \neg B)$ , we get  $\top \leq (\neg A \wedge \neg B)$ , by (P3), (E1) and (E2).<sup>13</sup> By (E2),  $A \vee \neg B$  and  $\neg A \vee B$  are also maximally entrenched. Since  $A$  is equivalent to  $(A \vee B) \wedge (A \vee \neg B)$ , we thus find, by (E1) – (E3), that  $A$  receives the same degree as  $A \vee B$ , i.e.,  $A \leq A \vee B$  as well as  $A \vee B \leq A$ . Since on the other hand  $B$  is equivalent to  $(A \vee B) \wedge (\neg A \vee B)$ , we get by the same reasoning that  $B$  receives the same degree as  $A \vee B$ , too. Hence  $A$  and  $B$  must have the same degrees. Since we can choose  $\perp$  for  $A$ , say, we find that all non-maximally entrenched sentences are in fact minimally entrenched. Thus the relation  $\leq$  is trivial in that it distinguishes at most two degrees of belief. We conclude that there are rather tight limits to the functionality of the degrees of belief – at least as long as we insist that sentences that are logically equivalent with respect to classical logic should receive the same degree.<sup>14</sup> We reject the disjunctiveness condition (P3) as a condition for entrenchments or degrees of *beliefs*. The degree of belief of a disjunction can definitely be higher than that of its disjuncts. Similarly, we reject the conjunctiveness condition for plausibilities or degrees of disbelief.

So there is a tension between degrees for beliefs and degrees for disbeliefs. The former are functional with respect to conjunctions, but cannot be so with respect to disjunctions, the latter have it just the other way round. Can we still piece together the relations ordering beliefs and the relations ordering disbeliefs

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<sup>12</sup>This result is contained in Rott (1991), but there are several related observations around, for instance in a notorious little paper by Elkan (1994). Seen from a multiple-valued logic perspective, it is (E2) (and also (E1)) that is not acceptable.

<sup>13</sup>If  $A$  and  $B$  were both *beliefs*, this result would be very strange indeed since the conjunction of the *negations* of two beliefs would turn out to be maximally entrenched.

<sup>14</sup>Which follows from (E2). Note that we haven’t talked about the functionality of negation in this argument. While we shall not be able to fix the rift between the functionality of ‘and’ and ‘or’, we shall install unrestricted functionality for negation.

in a reasonable way? It turns out that this is possible. The idea of combining entrenchment and plausibility (necessity and possibility) to a single scale has been first explored by Spohn (1991, p. 169; 2002, p. 378) and Rabinowicz (1995, pp. 111–112, 123–127).<sup>15</sup>

**4.1 Rabinowicz likelihood relations.** Rabinowicz studied relations that make beliefs and disbeliefs fully comparable. We change the numbering of Rabinowicz’s axioms in order to have a better correspondence with the relations we have seen so far.

- (L1) If  $A \leq_l B$  and  $B \leq_l C$  then  $A \leq_l C$  *Transitivity*
- (L2)  $A \leq_l B$  or  $B \leq_l A$  *Connectivity*
- (L3) If  $A \vdash B$  then  $A \leq_l B$  *Dominance*
- (L4) If  $\neg A <_l A$  and  $\neg B <_l B$ , then  $A \leq_l A \wedge B$  or  $B \leq_l A \wedge B$   
*Positive conjunctiveness*
- (L5) If  $A \leq_l B$  then  $\neg B \leq_l \neg A$  *Contraposition*

(L1) and (L3) parallel analogous conditions for entrenchment and plausibility. (L2) is needed since in contrast to the cases of entrenchment and plausibility, the connectivity of the relation  $\leq$  is no longer derivable from the other conditions. The validity of the conjunctiveness condition (E3) for entrenchment relations is restricted for likelihood relations to pairs sentences that are more likely than their negations (‘likely’ is here not meant in a probabilistic sense). Thus condition (L4).<sup>16</sup> Finally, there is a new condition (L5), called ‘contraposition’ by Rabinowicz, that takes care of negations.

As in the case of entrenchment and plausibility relations, we treat as optional two conditions concerning the maximum and minimum of plausibility.

- (L6)  $\neg A <_l A$  iff  $A$  is believed *Positivity*
- (L7) If  $\top \leq_l B$  then  $\vdash B$  *Maximality*

(L7) says that only tautologies are maximally likely, a condition that we do not want to endorse as universally valid. (L6) can be interpreted as a definition of the notion of *belief*. A sentence  $A$  is believed iff it is more likely than its negation, i.e., iff  $\neg A <_l A$ . Consequently,  $A$  is a disbelieved iff  $A < \neg A$ , and  $A$  is a non-belief iff both  $A \leq \neg A$  and  $\neg A \leq A$ . In likelihood relations, the belief

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<sup>15</sup>But see footnote 17.

<sup>16</sup>Rabinowicz’ original axiom is actually not (L4), but

(L4<sup>R</sup>) If  $\neg C <_l C$ ,  $C \leq_l A$  and  $C \leq_l B$ , then  $C \leq_l A \wedge B$ .

(L4<sup>R</sup>) can be proven equivalent to (L4) on the basis of the other axioms (the proof of this and other proofs are collected in Appendix 1 at the end of this paper). – Note also that on the basis of the other axioms, (L4) can equivalently be strengthened to

(L4<sup>+</sup>) If  $\neg A \leq_l A$  and  $\neg B <_l B$ , then  $A \leq_l A \wedge B$  or  $B \leq_l A \wedge B$ .

status can no longer be expressed as a relation between  $A$  and either  $\top$  or  $\perp$ . But notice that in any case,  $A$  is a belief if and only if  $\neg A < A$ , regardless of whether  $\leq$  is supposed to stand for  $\leq_e$ ,  $\leq_p$  or  $\leq_l$ .

By (L4), likelihood relations are functional with respect to conjunction for beliefs. Using contraposition, it is easy to see that they are functional with respect to disjunctions for disbeliefs. A natural question to ask is whether there is any functionality “across the categories”, for instance, when  $A$  is a belief and  $B$  is a disbelief. The following facts are derivable from the axioms (L1)–(L5). Roughly, they say that in this case, if  $A$  is more firmly believed than  $B$  is disbelieved, then  $A \wedge B$  is as likely as  $B$ , while if  $A$  is less firmly believed than  $B$  is disbelieved, then  $A \vee B$  is as likely as  $A$ .

(LC $\wedge$ ) If  $B \leq_l \neg B <_l A$ , then  $B \leq_l A \wedge B$

(LC $\vee$ ) If  $B <_l \neg A \leq_l A$ , then  $A \vee B \leq_l A$

Likelihood relations express distinctions between both beliefs and disbeliefs. However, all the sentences that are neither believed or disbelieved, i.e., that are non-believed in the narrow sense, get the same likelihood level ‘below’ all the beliefs and ‘above’ all the disbeliefs. As far as Rabinowicz likelihood is concerned, there is nothing to distinguish between them. Distinctions in likelihood are only made between sentences that are either believed or disbelieved.

Rabinowicz’ motivation for introducing likelihood relations is given by the following

**Observation 6 (Rabinowicz)** *Take an entrenchment relation  $\leq_e$ , and define the corresponding plausibility relation  $\leq_p$  as in Observation 2. Then the relation  $\leq_l$  defined by*

$$A \leq_l B \quad \text{iff} \quad \text{both } A \leq_e B \text{ and } A \leq_p B$$

*is a likelihood relation.*

Note that the definition of  $\leq_l$  from an entrenchment relation  $\leq_e$  and a plausibility relation  $\leq_p$  in Observation 6 produces a likelihood relation only if  $\leq_e$  and  $\leq_p$  fit together in the sense that they satisfy the condition  $A \leq_p B$  iff  $\neg B \leq_e \neg A$  (or, of course equivalently,  $A \leq_e B$  iff  $\neg B \leq_p \neg A$ ).

Rabinowicz also shows how to reconstruct the entrenchment relation (and thus, also the plausibility relation) corresponding to a given likelihood relation. He defines  $A \leq_e B$  if and only if  $A \leq_l B$  or  $A \leq_l \neg A$ . Using this definition, he is able to demonstrate formally that entrenchment and likelihood are “equivalent concepts” (1995, p. 125).

**4.2 Spohnian beta functions.** Now suppose an entrenchment ranking function  $\varepsilon$  is given, and  $\pi$  is its associated plausibility function as defined in Observation 3. We are looking for a numerical function that assigns ranks to both beliefs

and disbeliefs. Except for notational differences, the following suggestion is due to Wolfgang Spohn (1991, p. 169):<sup>17</sup>

$$\beta(A) = \varepsilon(A) + \pi(A)$$

Notice that  $\beta(A)$  equals  $\varepsilon(A)$  if  $\varepsilon(A)$  is positive, i.e., if  $A$  is believed, and equals  $\pi(A)$  otherwise. Because either  $\varepsilon(A)$  or  $\pi(A)$  is 0, we do not need “real” addition here, the plus sign is just a convenient notational device. Another way of conceiving of beta functions is viewing them as combining the pair of entrenchment and plausibility values by applying restricted maximum and minimum operations on them. This view will prove to be interesting later.

$$\beta(A) = \begin{cases} \max\{\varepsilon(A), \pi(A)\} & \text{if } \varepsilon(A) > 0 \\ \min\{\varepsilon(A), \pi(A)\} & \text{otherwise} \end{cases}$$

As far as I know, no axiomatic characterization of beta functions has been given yet. So let us propose one here. A *beta function*, or also *likelihood function*,<sup>18</sup>  $\beta$  assigns to each sentence an integer such that

- (Bi) Sentences equivalent under  $\vdash$  get the same  $\beta$ -value *Intensionality*
- (Bii)  $\beta(\top) \geq 0$  *Top*
- (Biii) If  $\beta(A) \geq 0$  and  $\beta(B) > 0$ , then  $\beta(A \wedge B) = \min\{\beta(A), \beta(B)\}$  *Positive conjunctiveness*
- (Biv)  $\beta(\neg A) = -\beta(A)$  *Inversion*

The inversion condition (Biv) for beta functions is the counterpart of the contraposition condition (L5) for likelihood relations. The positive conjunctiveness condition (Biii) strengthens its relational counterpart (L4).<sup>19</sup> As a consequence, the dominance condition is not needed as a separate axiom. It follows from (Bi)–(Biv) that if  $A \vdash B$  then  $\beta(A) \leq \beta(B)$ .

The following interpretation of beta functions was the one intended by Spohn:  $\beta(A)$  is positive (negative, or zero) if and only if  $A$  is believed (disbelieved or, respectively, a non-belief in the narrow sense). We will treat this interpretation as optional, but emphasize that beta functions in this interpretation distinguish

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<sup>17</sup>This concept of “firmness of belief” (and its relational projection “more plausible than”) was present already in Spohn (1988, p. 116). Spohn (1991) attributes the following elegant definition, or actually its notational variant  $\beta(A) = \kappa(\bar{A}) - \kappa(A)$ , to Bernard Walliser. Variants of a similar combination of necessity and possibility functions are mentioned and related to the certainty factors of MYCIN (Buchanan and Shortliffe 1984) by Dubois and Prade (1988a, p. 295; 1988b, pp. 246, 254), Dubois, Prade and Smets (1996, end of Section 4.1) and Dubois, Moral and Prade (1998, p. 349).

<sup>18</sup>Spohn calls such functions ‘belief functions’, but I want to avoid this term because (a) it is rather unspecific, and (b) in so far as it has an established meaning, it is commonly associated with the work of Arthur Dempster and Glenn Shafer.

<sup>19</sup>We pointed out in footnote 16 that the relational counterpart (L4<sup>+</sup>) of (Biii) is derivable from the axioms for likelihood relations.

ranks between both beliefs and disbeliefs. (Bii) says that tautologies must not be disbelieved. If  $\beta(A) \geq \beta(\top)$ , we say that  $A$  is a priori, and if  $\beta(A) \leq \beta(\perp)$ , then  $\neg A$  is a priori.

As in the relational case, we have functionality for conjunction among beliefs, for disjunction among disbeliefs, and the following restricted “cross-categorical” functionality of conjunction and disjunction:

(BC $\wedge$ ) If  $0 \leq -\beta(B) < \beta(A)$ , then  $\beta(A \wedge B) = \beta(B)$

(BC $\vee$ ) If  $0 \leq \beta(A) < -\beta(B)$ , then  $\beta(A \vee B) = \beta(A)$

It is easy to construct the entrenchment and plausibility functions corresponding to a given beta function. If we put  $\varepsilon(A) = \max\{\beta(A), 0\}$  and  $\pi(A) = -\varepsilon(\neg A) = -\max\{\beta(\neg A), 0\} = -\max\{-\beta(A), 0\} = \min\{\beta(A), 0\}$ , it can be proved that entrenchment and likelihood functions are “equivalent” concepts, as indeed are plausibility and likelihood functions. This justifies our claim that the above conditions axiomatically characterize Spohn’s idea of beta functions.

**Observation 7** *A function  $\beta$  is a likelihood function satisfying (Bi)–(Biv) if and only if there is an entrenchment function  $\varepsilon$  satisfying (Ei)–(Eiii) such that*

$$\beta(A) = \varepsilon(A) + \pi(A)$$

where  $\pi$  is the plausibility function corresponding to  $\varepsilon$ , defined by  $\pi(A) = -\varepsilon(\neg A)$ .

Spohn (2002, p. 378) argued that “belief functions [i.e., beta functions, HR] may appear to be more natural [than plausibility functions, HR]. But their formal behaviour is more awkward.”<sup>20</sup> For the purposes of the present paper with its focus on the concept of comparative degrees of belief, however, it is sufficient that *there are* systematic and well-understood non-probabilistic rankings of beliefs and disbeliefs along a single scale. We have axiomatized them and then identified a number of interesting facts about them. So perhaps they look a little less awkward now.<sup>21</sup> If one were still inclined to call their formal behaviour awkward, then this would not speak against the reasonableness of the notion of unified degrees of belief and disbelief.

### 4.3 Spohnian beta functions and Rabinowicz likelihood relations.

It turns out that the qualitative counterparts of Spohnian belief functions are exactly the Rabinowicz likelihood relations.

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<sup>20</sup>Dubois, Moral and Prade (1998, p. 349) gave a similar comment: “It turns out that this set-function is not easy to handle beyond binary universes, because it is not compositional whatsoever.”

<sup>21</sup>Perhaps as awkward as the cubic function  $y = x^3$  which is concave for  $x < 0$  and convex for  $x > 0$ .

**Observation 8** Take a Spohnian beta function  $\beta$ . Then its relational projection defined by

$$A \leq_l B \text{ iff } \beta(A) \leq \beta(B)$$

is a Rabinowicz likelihood relation.

Conversely, for every finite Rabinowicz likelihood relation  $\leq_l$  there is a Spohnian beta function  $\beta$  such that  $\leq_l$  is the relational projection of  $\beta$ .

Sentences mapped to zero by a beta function are exactly those that are as likely as their negations under the corresponding likelihood relation. For the proof of the second part of Observation 8, one takes the equivalence class of sentences that are as likely as their negations as the class of sentences getting rank zero by the beta function  $\beta$ . Then one assigns numbers to all other equivalence classes with respect to  $\simeq_l$  going up and going down from zero according to the ordering relation induced by  $\leq_l$ . Due to contraposition (L5), everything happening in the negative integers will be perfectly symmetrical to what goes on in the positive integers.

Let us now give a graphical illustration of the situation so far. The model for belief states most easily comprehended is Grove's (1988) subjectivist variant of Lewis's (1973) objectivist conception of *system of spheres*. It represents a doxastic state by a system of nested sets of possible worlds.<sup>22</sup> The smallest set "in the center" is the set of possible worlds which the reasoner believes to contain the actual world  $w_a$ , i.e., the worlds considered "possible" according to the reasoner's beliefs. If she receives evidence that the actual world is not contained in this smallest set, she falls back on the next larger set of possible worlds. Thus the first shell<sup>23</sup> around the center contains the worlds considered second most plausible by the reasoner. And again, should it turn out that the actual world is not to be found in this set either, the reasoner is prepared to fall back on her next larger set of possible worlds. And so on. The sets or *spheres* of possible worlds correspond to grades of plausibility, or to put it differently, grades of deviation from the subject's actual beliefs. The system of spheres taken as a whole can be thought of as representing a person's *mental* or, more precisely, her *doxastic state*.<sup>24</sup>

How can we use this modelling to codify the degrees of belief and disbelief (entrenchment or plausibility) of a given sentence? Such degrees are determined by the sets of spheres throughout which this sentence holds universally, and the

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<sup>22</sup>For the sake of simplicity, I suppose that all sets mentioned in this model are finite. Technically, the "worlds" should be thought of as the models of a finitary propositional language.

<sup>23</sup>A *shell* is the difference set between two neighbouring spheres. Spheres are nested, shells are disjoint.

<sup>24</sup>Of course, it must not be expected that the system of spheres is centered on a single world  $w_a$  that represents the actual world. The facts that (i) the innermost circle need not be a singleton and (ii) it need not contain the actual world distinguish Grove's subjectivist from Lewis' objectivist system-of-spheres model. If one of the reasoner's beliefs is wrong, then  $w_a$  is not contained in the innermost sphere. It may indeed be located at any arbitrary position in the sphere system.

sets of spheres which it intersects. If  $A$  covers more spheres than  $B$ , then  $A$  is more entrenched than  $B$ . If  $A$  intersects more spheres than  $B$ , then  $A$  is more plausible than  $B$ . Fig. 1 gives an illustration of the degrees of belief of three sentences  $A$ ,  $B$  and  $C$  in a doxastic state represented by a Grovean system of spheres. I remind the reader that the numerals in the qualitative approach are not supposed to denote genuine numbers, they are just used to indicate the relative positions in a weak total ordering. There is no sphere labelled ‘0’ in Fig. 1. That  $\beta(B) = 0$  means that the innermost sphere (labelled ‘1’) contains both  $B$ -worlds and  $\neg B$ -worlds.

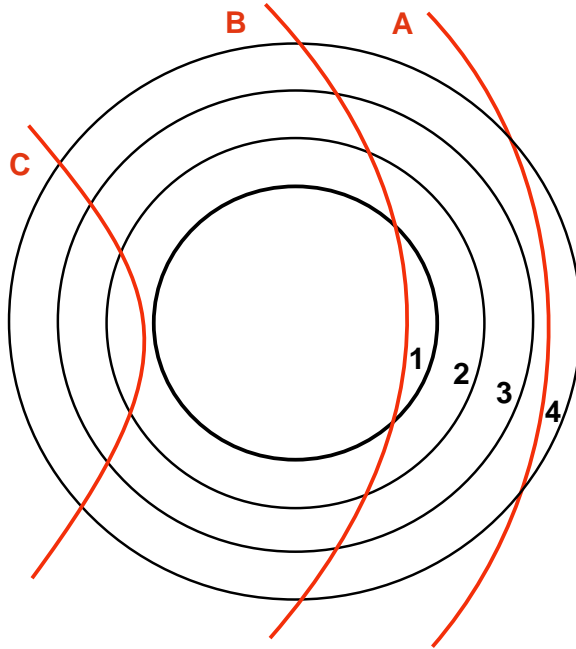


Fig. 1: Degrees of belief and disbelief:  $\beta(A) = 3$ ,  $\beta(B) = 0$  and  $\beta(C) = -1$ .  
 $A$  is a belief,  $B$  a non-belief, and  $C$  a disbelief.

## 5 DEGREES FOR NON-BELIEFS: EXPECTATIONS, DISEXPECTATIONS, NON-EXPECTATIONS

We have seen that we can, drawing on the work of Rabinowicz and Spohn, map degrees of beliefs and disbeliefs into a single dimension in a reasonable way that ranks disbeliefs lower than beliefs. But usually there are lots of things that a reasoner is ignorant of, honestly most reasoners would have to admit that they neither believe nor disbelieve most of the propositions they could be asked about. Yet all of these myriads of non-beliefs in the narrow sense are mapped by a beta function onto a single zero point. Intuitively, however, there can be



vast differences between the credibilities of various non-beliefs. Some of them are considered to be quite likely, while others would be found very surprising. We should like to be able to express such differentiations. What should we do then with the non-beliefs?

One perfectly good way of proceeding would be to use probabilities in order to express the different doxastic attitudes toward non-beliefs.<sup>25</sup> The fact that probability distributions are not functional with respect to either  $\wedge$  and  $\vee$ , however, makes it evident that the introduction of probabilities makes the model a hybrid. Although this is not a decisive argument against using probabilities, it would be nicer if we continued with our ‘logical’ approach and distinguished among non-beliefs in terms of comparative necessity and possibility in just the way in which we have assigned degrees to beliefs and disbeliefs. Would it make sense to stipulate an expectation ordering of non-beliefs analogous to the entrenchment ordering of beliefs, perhaps in such a way that a selected set of expectations is logically closed and consistent? It turns out that this is indeed possible. We can combine entrenchment and plausibility structures for beliefs and disbeliefs with similar structures for non-beliefs. Intuitively, we just need to fan out the zero point of likelihood relations and functions into a multitude of different ranks.

The key idea is this. Reasoners do not only have beliefs, but also things they *almost* believe, or things they *would* believe if they were just a little bolder than they are: They have opinions, expectations and hypotheses, they make conjectures and default assumptions, they act on presumptions etc. It is not necessary to decide here which of these pro-attitudes are stronger than which of the others. The point is that by increasing their degree of boldness (or degree of credulity, gullibility etc.), reasoners can successively strengthen the set of accepted sentences, until they reach a point at which they refuse to further raise their credulity. A sequence of successively increasing ‘expectation sets’ emerges.<sup>26</sup> Following the lead of Gärdenfors and Makinson (1994), I use *expectation* as the generic term for such “subdoxastic” attitudes. Let us call *weak expectations* the sentences accepted at the maximum level of boldness. Semantically, this process of stepwise extending one’s expectation set means establishing an inverted Grove model. Starting with the set of worlds that represent the reasoner’s beliefs, new spheres that go inward are added. Each such set of worlds corresponds to a member of the sequence of the reasoner’s gradually enlarged expectation sets. In this system of spheres, the outermost sphere represents the reasoner’s belief, and the innermost sphere represents her weakest expectations, that is, the sentences she is ready to accept at her highest level of doxastic boldness.

Having established ‘inner spheres’ of expectation, some non-beliefs turn out

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<sup>25</sup>The most well-known way of having a probability distribution within each ordinal rank has emerged in the research related to so-called Popper measures, cf. van Fraassen (1976), Spohn (1986), Hammond (1994) and Halpern (2001).

<sup>26</sup>We assume for simplicity that this sequence is finite.

to be comparatively plausible and positively expected to some degree. However, other non-beliefs will turn out to be implausible and surprising (or ‘disexpected’) to some degree. Some non-beliefs will be neither expected nor disexpected, namely those that are neither implied nor contradicted by the boldest theory entertained by the reasoner.

**5.1 Relations for non-beliefs.** Interestingly, one can use precisely the same relational structures of comparative necessity and possibility for evaluating expectations as we employed for beliefs. We can re-use our axioms (E1)–(E3), (P1)–(P3) and (L1)–(L5) as before, but for the sake of clarity we rename them into (Ex1)–(Ex3), (Px1)–(Px3) and (Lx1)–(Lx5) in the present context, and apply the relation symbols  $\leq_{ex}$ ,  $\leq_{px}$  and  $\leq_{lx}$ . All these relations are total.

The only difference lies in the interpretation. The optional conditions (E4) and (E5) and their counterparts for plausibility and likelihood are no longer appropriate. They should be replaced by

(Ex4)  $\perp <_{ex} A$  iff  $A$  is weakly expected

(Ex5)  $\top \leq_{ex} A$  iff  $A$  is believed

(Px4)  $A <_{px} \top$  iff  $\neg A$  is weakly expected

(Px5)  $A \leq_{px} \perp$  iff  $\neg A$  is believed

(Lx6)  $\neg A <_{lx} A$  iff  $A$  is weakly expected

(Lx7)  $\top \leq_{lx} A$  iff  $A$  is believed

Expectation relations  $\leq_{ex}$  are similar to the relations with the same name introduced by Gärdenfors and Makinson (1994). Expectation plausibility relations  $\leq_{px}$  establish comparisons of ‘disexpectations’ (with full comparability). I am not aware that they have been presented in this way in the literature, but they are straightforward to introduce on our account.<sup>27</sup>

Notice that  $A$  is weakly expected if and only if  $\neg A <_x A$ , regardless of whether  $\leq_x$  is supposed to stand for  $\leq_{ex}$ ,  $\leq_{px}$  or  $\leq_{lx}$ .

As in the case of belief and disbelief, we can say that the relevant relations *fit together* if, for example, they satisfy the duality principle  $A \leq_{px} B$  iff  $\neg B \leq_{ex} \neg A$ , or the condition  $A \leq_{lx} B$  iff  $A \leq_{ex} B$  and  $A \leq_{px} B$ , or the condition  $A \leq_{ex} B$  iff  $A \leq_{lx} B$  or  $A \leq_{lx} \neg A$ .

Focussing on the comparative necessity structures, the only difference between entrenchment and expectation relations is that beliefs occupy the (usually: many) non-minimal ranks in entrenchment relations, while they occupy the (single) maximal rank in expectation relations. By the same token, non-beliefs occupy the (single) minimal rank in entrenchment relations, while they occupy

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<sup>27</sup>I shall continue to use the artificial term ‘disexpectation’ as short for ‘expectation-that-not’. If neither  $A$  nor  $\neg A$  is expected, I will call  $A$  a ‘non-expectation’.

the (usually: many) non-maximal ranks in expectation relations.

The relations are intertwined in a way similar to the corresponding relations for beliefs and disbeliefs

**Observation 9** (i) *Take an expectation relation  $\leq_{ex}$ . Then its dual defined by*

$$A \leq_{px} B \text{ iff } \neg B \leq_{ex} \neg A$$

*is a plausibility relation for expectations. And vice versa.*

(ii) *Take an expectation relation  $\leq_{ex}$  and define the corresponding plausibility relation  $\leq_{px}$  for expectations. Then the relation defined by*

$$A \leq_{lx} B \text{ iff both } A \leq_{ex} B \text{ and } A \leq_{px} B$$

*is a likelihood relation for expectations.*

*Conversely, take a likelihood relation  $\leq_{ex}$  for expectations. Then the relation defined by*

$$A \leq_{ex} B \text{ iff both } A \leq_{lx} B \text{ or } A \leq_{lx} \neg A$$

*is the corresponding expectation relation.*

While expectation relations are comparative necessity relations, plausibility relations for expectations are comparative possibility relations. Likelihood relations for expectations have the same hybrid structure as the likelihood relations for beliefs.

**5.2 Functions for non-beliefs.** Can we design ‘expectation functions’ analogous to belief and disbelief functions, just with different ‘limiting cases’? Yes, we can. The intuitive idea is that we use 1 as the threshold value for belief and -1 as the threshold value for disbelief. Non-beliefs receive degrees between the degrees of beliefs and the degrees of disbeliefs, that is, between -1 and +1. We will use inverse integers for this task. However, the reader should be warned once more that the numerals are not supposed to represent anything more than the relative positions in a weak total ordering. In particular, the distance between  $\frac{1}{2}$  and  $\frac{1}{3}$ , say, is not meant to be smaller than the distance between 2 and 3. Both pairs signify neighbouring ranks. And of course fractions such as  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}, \dots$  should not be mistaken for probabilities. As before, we assume that the range of values of all functions that follow is finite.

An *expectation ranking function*, or simply *expectation function*,  $\varepsilon_x$  assigns to each sentence an inverse positive integer 1,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}, \dots$  or 0 in such a way that (Ei)–(Eiii) are satisfied, with the understanding that  $\varepsilon_x(A) = 1$  means that  $A$  is believed, and  $\varepsilon_x(A) > 0$  means that  $A$  is weakly expected by a reasoner with expectation state  $\varepsilon_x$ .

A *plausibility ranking function*, or simply *plausibility function*, for expectations  $\pi_x$  assigns to each sentence a negative inverse integer  $-1$ ,  $-\frac{1}{2}$ ,  $-\frac{1}{3}$ ,

$-1/4, \dots$  or 0 in such a way (Pi)–(Piii) are satisfied, with the understanding that  $\pi_x(A) = -1$  means that  $\neg A$  is believed, and  $\pi_x(A) < 0$  means that  $\neg A$  is weakly expected by a reasoner with expectation state  $\pi_x$ .

A *likelihood function for expectations*,  $\beta_x$  assigns to each sentence an inverse integer  $\pm 1, \pm 1/2, \pm 1/3, \pm 1/4, \dots$  or 0 in such a way (Bi)–(Biv) are satisfied, with the understanding that  $\beta_x(A) = 1$  means that  $A$  is believed, and  $\beta_x(A) = 0$  means that neither  $A$  nor  $\neg A$  is weakly expected by a reasoner with expectation state  $\beta_x$ .

Notice that the sentences that receive value 0 by the functions  $\varepsilon$ ,  $\pi$  and  $\beta$ , i.e., the non-beliefs in the narrow sense, can now be differentiated by assigning to them (finitely many) values lying in the interval  $[-1/2, 1/2]$ . This provides an enormous resource of fine-grained degrees for non-beliefs. On the other hand, functions for non-beliefs do not report any distinctions between beliefs or distinctions between disbeliefs. Fig. 2 gives an example in system-of-spheres representation. Note that this time, the reasoner’s beliefs are represented by the outermost sphere, rather than by the innermost sphere as in Fig. 1. Again, there is no sphere labelled ‘0’, a  $\beta$ -value of zero means that the proposition in question intersects but does not cover the innermost sphere (labelled ‘ $1/4$ ’ in Fig. 2).

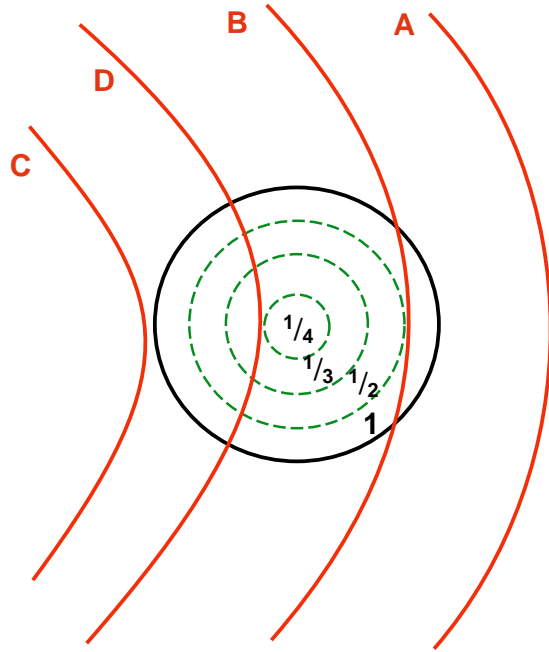


Fig. 2: Degrees of expectation and disexpectation (= degrees of non-belief):  $\beta_x(A) = 1$ ,  $\beta_x(B) = 1/2$ ,  $\beta_x(D) = -1/4$ ,  $\beta_x(C) = -1$   
 $A$  is a belief,  $B$  an expectation,  $C$  a disbelief, and  $D$  is a disexpectation.

## 6 COMBINING DEGREES OF BELIEFS AND DISBELIEFS WITH DEGREES FOR NON-BELIEFS

We have treated expectations formally exactly like beliefs – except that we indicated that they are not quite beliefs, but strictly speaking non-beliefs. Gärdenfors and Makinson have rightly pointed out that also from an intuitive point of view, beliefs and expectations are not so different after all:

Epistemologically, the difference between belief sets and expectations lies only in our attitude to them, i.e., what we are willing to do with them. For so long as we are *using* a belief set  $K$ , its elements function as full beliefs. But as soon as we seek to *revise*  $K$ , thus putting its elements into question, they lose the status of full belief and become merely expectations, some of which may have to go in order to make consistent place for beliefs introduced in the revision process. (Gärdenfors and Makinson 1994, pp. 223-224)

Gärdenfors and Makinson did much to uncover the analogy between belief structures and expectation structures in their seminal papers (Makinson and Gärdenfors 1991, Gärdenfors and Makinson 1994), but they did not unify beliefs and expectations into an all-encompassing doxastic state.

Semantically, what has to be done in order to get beliefs and expectations into a joint representation is quite clear. One just has to superimpose the outward-directed system of spheres for beliefs and disbeliefs on the inward-directed system of spheres for non-beliefs (see Fig. 3 for illustration). The only precondition for this operation to succeed is that the two systems of spheres *fit together*. The innermost sphere of the former must be identical with the outermost sphere of the latter: These spheres are both supposed to represent the reasoner's beliefs. What we have to do now is to transfer this pictorial description to our various relations and functions representing degrees of belief.

**6.1 Combining relations for beliefs and disbeliefs with relations for non-beliefs.** Before we can start merging degrees for beliefs and disbeliefs with degrees for non-beliefs, we need to make sure that the relevant orderings *fit together*. This is the case if the beliefs marked out by the relation  $\leq_e$  (or by  $\leq_p$  or by  $\leq_l$ ) are identical with the beliefs marked out by the relation  $\leq_{ex}$  (or respectively, by  $\leq_{px}$  or by  $\leq_{lx}$ ). Thus, when joining comparative necessity, comparative possibility and comparative likelihoods, we require that the following *fitting conditions* are satisfied for all sentences  $A$ :

- (i)  $\perp <_e A$  iff  $\top \leq_{ex} A$
- (ii)  $A <_p \top$  iff  $A \leq_{px} \perp$
- (iii)  $\neg A <_l A$  iff  $\top \leq_{lx} A$

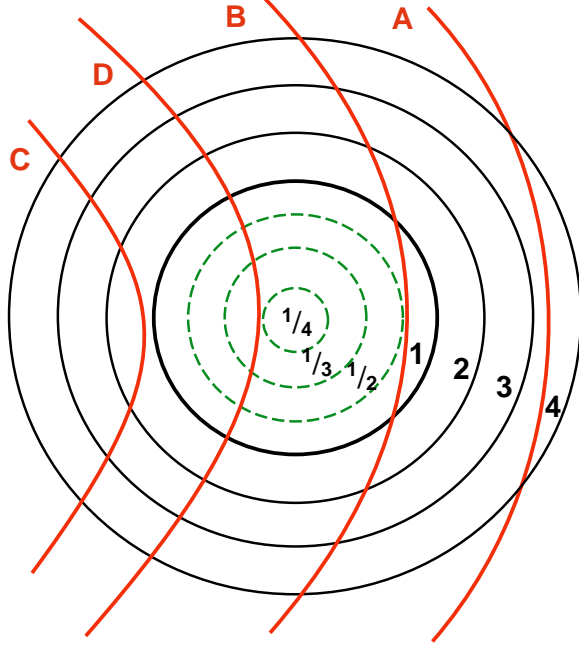


Fig. 3: Degrees of belief, non-belief and disbelief:  
 $\beta_{all}(A) = 3$ ,  $\beta_{all}(B) = 1/2$ ,  $\beta_{all}(D) = -1/4$ ,  $\beta_{all}(C) = -1$

In the following, we use the relation symbols  $\leq_{ee}$ ,  $\leq_{pp}$  and  $\leq_{ll}$  for the relations that combine the respective relations for beliefs/disbeliefs and non-beliefs.

If the relevant belief sets coincide, then there is no reason why entrenchment and expectation relations should not be merged into a single homogeneous *comparative necessity relation* satisfying (E1)–(E3).

$$A \leq_{ee} B \text{ iff } A \leq_e B \text{ and } A \leq_{ex} B$$

The transitivity and dominance conditions for  $\leq_e$  and  $\leq_{ex}$  transfer immediately to  $\leq_{ee}$ . With the help of the ‘fitting condition’ (i), one can also show that  $\leq_{ee}$  satisfies conjunctiveness. The maxima of the combined relation  $\leq_{ee}$  are the a priori beliefs, the minima are those propositions that are not even weak expectations.

The plausibility relations concerning disbeliefs and non-beliefs (in the narrow sense) can similarly be combined into a homogeneous *comparative possibility relation* satisfying (P1)–(P3).

$$A \leq_{pp} B \text{ iff } A \leq_p B \text{ and } A \leq_{px} B$$

The transitivity and dominance conditions for  $\leq_p$  and  $\leq_{px}$  transfer immediately to  $\leq_{pp}$ . With the help of the fitting condition (ii), one can also show that  $\leq_{pp}$  satisfies disjunctiveness.

Finally, the likelihood relations concerning beliefs/disbeliefs and non-beliefs (in the narrow sense) can be combined similarly into a homogeneous *comparative likelihood relation* satisfying (L1)–(L5).

$$A \leq_l B \text{ iff } A \leq_l B \text{ and } A \leq_{lx} B$$

The relation  $\leq_l$  is the most comprehensive or fine-grained notion of degree that we have: It draws distinctions in degree between beliefs and disbeliefs and non-beliefs, with the latter in turn split up into expectations, disexpectations and non-expectations.

**6.2 Combining functions for beliefs and disbeliefs with functions for non-beliefs.** It will come to no surprise that we can achieve an analogous unification with functions rather than relations. For the merger to succeed, the beliefs marked out by the function  $\varepsilon$  (or by the functions  $\pi$  and  $\beta$ ) must fit together with the beliefs marked out by the function  $\varepsilon_x$  (or, respectively, by the functions  $\pi_x$  and  $\beta_x$ ). When joining comparative necessity, comparative possibility and comparative likelihoods, we thus require that for all  $A$

- (i)  $\varepsilon(A) \geq 1$  iff  $\varepsilon_x(A) \geq 1$
- (ii)  $\pi(A) \leq -1$  iff  $\pi_x(A) \leq -1$
- (iii)  $\beta(A) \geq 1$  iff  $\beta_x(A) \geq 1$ , and  $\beta(A) \leq -1$  iff  $\beta_x(A) \leq -1$

If the relevant functions fit together, then we can again combine them into functions specifying all-encompassing degrees of belief, disbelief and non-belief. We use the function symbols  $\varepsilon_{all}$ ,  $\pi_{all}$  and  $\beta_{all}$  to denote them.

- (i)  $\varepsilon_{all}(A) = \max\{\varepsilon(A), \varepsilon_x(A)\}$
- (ii)  $\pi_{all}(A) = \min\{\pi(A), \pi_x(A)\}$
- (iii)  $\beta_{all}(A) = \begin{cases} \max\{\beta(A), \beta_x(A)\} & \text{if } \beta_x(A) > 0 \\ \min\{\beta(A), \beta_x(A)\} & \text{otherwise} \end{cases}$

It is easy to verify that the overall function  $\varepsilon_{all}$  is a *necessity function* that assigns to each sentence a non-negative integer or a positive inverse integer such that (Ei)–(Eiii) are satisfied; that  $\pi_{all}$  is a *possibility function* that assigns to each sentence a non-positive integer or a negative inverse integer such that (Pi)–(Piii) are satisfied; and that  $\beta_{all}$  is a *likelihood function*  $\beta_{all}$  that assigns to each sentence an integer or an inverse integer such that (Bi)–(Biv) are satisfied. Fig. 4 illustrates how the discrete degrees of belief are arranged along a line.

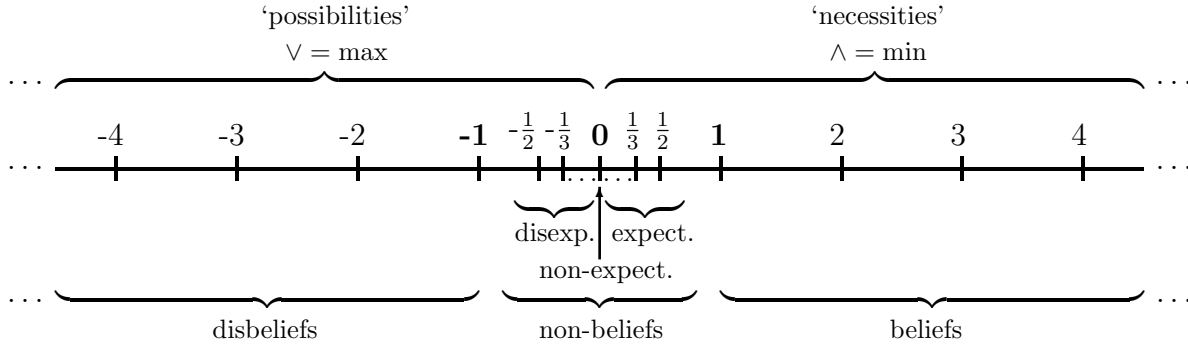


Fig. 4: Degrees of belief plotted on a line

## 7 LEVI ON DEGREES OF BELIEF AND DEGREES OF INCORRIGIBILITY

The British economist G.L.S. Shackle (1949, appendix; 1961, Chapter X) was perhaps the first person to introduce plausibility functions for expectations (under the name ‘degrees of potential surprise’) and also to consider expectation functions (under the name ‘degrees of belief’).<sup>28</sup> Isaac Levi picked up on Shackle’s work,<sup>29</sup> and has developed a sophisticated theory of his own that combines aspects related to beliefs and disbeliefs with aspects related to non-beliefs (or expectations). This is expressed, for instance, in Levi (1996, p. 267):

Shackle measures can be interpreted in at least two useful ways: in terms of caution- and partition-dependent deductively cogent inductive expansion rules and in terms of damped informational value of contraction strategies. One interpretation (as an assessment of incorrigibility) plays a role in characterizing optimal contractions . . . . The other interpretation plays a role in characterizing inductively extended expansions . . . . Although the formal structures are the same, the applications are clearly different.

For four decades, Levi has studied both expansions and contractions of sets of belief (‘corpora’ in his terminology). In a *contraction*, the reasoner gives up some specific sentence and makes use of the outward-directed spheres. These spheres are fallback positions for the case when a specified belief is to be withdrawn. The sort of expansions mainly considered by Levi, however, does not need any input of

<sup>28</sup>This is so in Levi’s streamlined accounts of the matter. Actually, Shackle struggled a lot with his notion of degree of surprise, and he certainly did not have a worked-out concept of degree of belief. As pointed out by Levi (1984, note 5) himself, Shackle was not quite consistent in his use of the term ‘degree of belief’. Taking Shackle’s potential surprise to be the function  $-\pi$ , we can recast his first official axiom as identifying the degree of belief in  $A$  with the pair  $\langle -\pi(A), -\pi(\neg A) \rangle$  while only on p. 71 in Shackle (1961) he determines the degree of belief as the value  $-\pi(\neg A) = \varepsilon(A)$ .

<sup>29</sup>Cf. Levi 1967, pp. 135–138; 1984; 1996, pp. 180–182; 2004, pp. 90–92.



a specific sentence. An *inductive expansion* aims at inductively enlarging a certain set of beliefs, making use of inward-directed spheres as bridgeheads for more daring inferential leaps. How far into unknown territory the reasoner advances depends on her boldness. No external instigation is needed to inductively expand a belief set. Levi gives a decision-theoretic derivation of the expected value of accepting a given sentence, with a degree of boldness serving as a parameter that tunes the comparative utilities of freedom-from-error and acquirement-of-new-information. The degree of expectation of a given sentence varies inversely with the degree of boldness that is needed in order to render that sentence acceptable.

In Levi's work, entrenchment relations (corresponding to outward-directed systems of spheres) characterize "degrees of incorrigibility."<sup>30</sup> Expectation relations are derived by Levi by allowing different degrees of "boldness" in inductive acceptance rules, and are on various occasions called "degrees of confidence of acceptance" (Levi's original term used in 1967), "degrees of belief", "degrees of certainty" or "degrees of plausibility." *Structurally*, degrees of incorrigibility and degrees of certainty are the same, in so far as they both obey the minimum-rule for conjunctions. The crucial difference is that a sentence *A* is believed if and only if it has a non-minimal degree of *incorrigibility*, and if and only if it has maximal degree of *certainty*.<sup>31</sup>

I think that by considering degrees of belief and disbelief along with degrees of non-belief, our model may also help to make transparent Levi's long insistence, which many have found hard to comprehend, that certainty (the feeling of infallibility) and incorrigibility are entirely different notions.<sup>32</sup> Degrees of certainty are needed for the construction of inductive expansions, degrees of incorrigibility are needed for the construction of belief contractions.<sup>33</sup>

The full scope of degrees of belief, disbeliefs and non-beliefs offered in this paper may help dissolving some misunderstandings that have haunted the literature for some time. For instance, the notion of 'plain belief' is used differently by Isaac Levi (2004, pp. 93–95, 179–180; 2006) and Wolfgang Spohn (1990; 2006).

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<sup>30</sup>As studied by Levi in his work on contractions, cf. Levi (1996, p. 264, 2004, p. 196). I take the liberty of glossing over the subtler differences that Levi (2004, pp. 191–199) identifies between degrees of entrenchment and degrees of incorrigibility.

<sup>31</sup>I do not claim it is easy to find this explicitly stated in Levi's writings. But it gets clear on contrasting the axioms for Shackle-style *b*-functions in Levi (1996, p. 181; 2004, p. 90) with the axioms for Gärdenfors-style *en*-functions in Levi (1996, p. 264; 2004, p. 198) for which the above description is entirely correct, if 'being believed' is identified with 'being contained in the corpus *K*'. Levi's *en*-functions are our  $\varepsilon$ -functions, while his *b*-functions are our  $\varepsilon_x$ -functions. Like Gärdenfors and Makinson, Levi has no counterparts to  $\varepsilon_{all}$ -functions.

<sup>32</sup>See Levi (1980, pp. 13–19, 58–62; 1991, pp. 141–146; 1996, pp. 261–268).

<sup>33</sup>Another important point to note is that while Levi uses both informational value and the probability of error for belief expansions, he only uses (damped) informational value as a criterion for belief contractions. Levi argues that belief contractions cannot incur any error to a person's belief system. That is certainly right, but error might be *removed* by a contraction. Levi's pragmatist philosophy, however, has no room for this, since in his picture a reasoner is invariably committed to the infallibility of her beliefs. For details, see Rott (2006).

What we have called simply ‘belief’ in this paper is, I think, called ‘full belief’ by Levi and ‘plain belief’ (or ‘belief simpliciter’) by Spohn. For Levi, plain belief is the sort of belief that would be reached after performing the inductive expansion recommended by epistemic decision theory.<sup>34</sup> In my terminology, this is an expectation to a certain degree. Some misunderstandings in the discussion may have arisen from the fact that most people have had in mind entrenchment or plausibility relations where Levi was thinking of degrees of non-belief or expectation. To my knowledge, neither Levi nor any of his critics has combined degrees of belief, disbelief and non-belief (expectation) into a single linear structure.<sup>35</sup>

## 8 CONCLUSION: ELUSIVE BELIEF

In many approaches of belief change and non-monotonic reasoning, researchers have used either degrees of belief or degrees of disbelief or degrees of non-belief (degrees of expectations). I have attempted to combine these various structures into a unified whole in this paper. The combination was achieved in two steps. In the first step, degrees of belief (necessity structures) were joined with degrees of disbelief (possibility structures). In the second step this combined structure was joined with a similar structure for (dis-)expectations rather than (dis-)beliefs, where this new expectation structure fans out a single point of the old belief structure, namely the zero point assigned to all the non-beliefs. For the first step, we were guided by work of Rabinowicz and Spohn, for the second step, we expanded on some thoughts of Levi.

I have said nothing about the application of this unified structure to belief change or non-monotonic reasoning tasks, because this seems rather straightforward: Just utilize that part of the ranking structure that is needed, and apply the well-known recipes. The point of the paper is, anyway, the idea of ‘degrees of belief’ itself.<sup>36</sup>

It appears that the structural difference between necessity relations and functions (representing belief and expectation) on the one hand, and possibility relations and functions (representing disbelief and potential surprise) on the other hand is more fundamental than the distinction between belief and expectations. After all, we have seen that there is no problem in defining all-encompassing necessity structures which have weak expectations (mere hypotheses, guesses, conjectures, etc.) occupying the lowest ranks and very strong, ineradicable be-

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<sup>34</sup>If the reasoner *actually* performs such an expansion, then, according to Levi, she in fact converts her plain beliefs into full beliefs.

<sup>35</sup>Levi (2004, p. 201) explicitly recommends against making this move.

<sup>36</sup>For more related structures and their applications to belief revision and non-monotonic reasoning, see Friedman and Halpern (1995, 2001) and Halpern (2003). These works are representative of excellent AI research of great technical sophistication. However, they do not follow through the idea of combining degrees of belief, non-belief and disbelief into a single scale. Their terminologies differ from the one used here.

liefs (that I called ‘a priori’) occupying the highest ranks. Structurally, there are no differences from the top to the bottom. Indeed, this explains the structural similarity between belief revision and non-monotonic reasoning first noted by Makinson and Gärdenfors (1991). But there is an essential structural contrast between the necessity structures of positive doxastic attitudes (belief, expectation) and the possibility structures of negative doxastic attitudes (disbelief, disexpectation), in that the latter obey a disjunction rule rather than a conjunction rule.

It is interesting that even though the various structures we used for encoding degrees of doxastic attitudes are not themselves uniform, the operations we used for merging them have turned out to be uniform. The combination of the relevant relations is always achieved by a conjunction:  $A \leq_{combined} B$  iff both  $A \leq_1 B$  and  $A \leq_2 B$ . The combination of the relevant functions can always be represented as a minimum-maximum-operation:  $f_{combined}(A)$  equals  $\max\{f_1(A), f_2(A)\}$  in some circumstances, and  $\min\{f_1(A), f_2(A)\}$  in others.

We pointed out in the introduction that some proponents of non-probabilistic approaches to the representation of belief states argued that a main advantage of their approaches is that they allow for the notion of ‘plain belief’ (or ‘belief simpliciter’) and ‘plain non-belief’ (or ‘non-belief simpliciter’). Probabilistic models are committed to assigning some number to any given proposition, and introspectively, this is just not what we feel like doing when we say that we believe something (or that we don’t believe it). Another advantage of non-probabilistic approaches is that they are not troubled by the failure of closure under conjunction that afflicts the high-probability interpretation to belief.

Richard Foley (1992, p. 111) interprets Locke’s (1690, Bk. IV, Ch. xv–xvi) discussion of probability and degrees of assent as warranting the

idea that belief-talk is a simple way of categorizing our degree of confidence in the truth of a proposition. To say that we believe a proposition is just to say that we are sufficiently confident of its truth for our attitude to be one of belief. Then it is epistemically rational for us to believe a proposition just in case it is epistemically rational for us to have sufficiently high degree of confidence in it, sufficiently high to make our attitude towards it one of belief.

Foley calls this idea *the Lockean Thesis*, and proposes that rational belief should be identified with a rational degree of confidence above some threshold level that the agent deems sufficient for belief. The lottery has taught us that it is difficult to reconcile this idea in probabilistic models of belief.<sup>37</sup> The (non-trivial) necessity structures that we have discussed in this paper do not have the problems afflicting high-probability approaches to belief. They guarantee that the sets of sentences above some specified threshold are all logically closed and consistent.

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<sup>37</sup>This ‘Lockean thesis’ has recently been probed in a probabilistic setting by Hawthorne and Bovens (1999) and Wheeler (2005).

So it seems that qualitative theories keep their promise of supplying a clear account of plain belief (and thus, of plain non-belief). The situation, however, is more complicated than that. In our final, all-encompassing comparative necessity relations  $\leq_{ee}$  and necessity functions  $\varepsilon_{all}$ , we have weak expectations, something like mere guesses at the bottom, and these are clear cases of non-belief. At the top we have a priori beliefs, which are clear cases of belief. Somewhere between the reasoner's expectations and a priori beliefs, her attitude must begin to be one of belief. But where to draw the line? Once entrenchment functions have been merged with expectation functions, the divide at the number 1 in the functional case of  $\varepsilon_{all}$  seems arbitrary, and in the relational case of  $\leq_{ee}$ , there is no dividing line to be found at all. Belief is a vague notion, and the threshold, if there really is one, is certainly context-dependent. We would set the threshold high in the courtroom interrogation, and we would set it low in a casual chat over lunch. There is no plain notion of belief. Accordingly, even though qualitative approaches to belief possess some advantages over probabilistic ones (they certainly possess some disadvantages, too), they do not single out a unique, clear and distinct notion of belief simpliciter. This is as it should be. Belief remains elusive.

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## APPENDIX I: SOME PROOFS

### A few little lemmas for likelihood relations

(a) Define  $A <_l B$  as the conjunction of  $A \leq_l B$  and  $B \not\leq_l A$ . Then transitivity for  $\leq_l$  implies:

If  $A \leq_l B$  and  $B <_l C$ , then  $A <_l C$ .

Proof. Let  $A \leq_l B$  and  $B <_l C$ .  $A \leq_l C$  follows from the transitivity of  $\leq$ . Suppose for reductio that  $C \leq_l A$ . Then by transitivity  $C \leq_l B$ , contradicting  $B <_l C$ .

If  $A <_l B$  and  $B \leq_l C$ , then  $A <_l C$ .

Proof. Similar

If  $A <_l B$  and  $B <_l C$ , then  $A <_l C$ .

Proof. Immediate consequence of the last two lemmas.

(b) If  $\neg A \leq_l A$  and  $A \leq_l B$ , then  $\neg B \leq_l B$

Proof: Let  $\neg A \leq_l A$  and  $A \leq_l B$ . From the latter, by contraposition  $\neg B \leq_l \neg A$ . So  $\neg B \leq_l \neg A \leq_l A \leq_l B$ , and by transitivity  $\neg B \leq_l B$ .

(c) If  $\neg A <_l A$  and  $A \leq_l B$ , then  $\neg B <_l B$

Proof: Let  $\neg A <_l A$  and  $A \leq_l B$ . From the latter, by contraposition  $\neg B \leq_l \neg A$ . So  $\neg B \leq_l \neg A <_l A \leq_l B$ , and by transitivity  $\neg B <_l B$ .

(d) If  $\neg A \leq_l A$  and  $A <_l B$ , then  $\neg B <_l B$

Proof: Let  $\neg A \leq_l A$  and  $A <_l B$ . From the latter, by contraposition  $\neg B <_l \neg A$ . So  $\neg B <_l \neg A \leq_l A <_l B$ , and by transitivity  $\neg B <_l B$ .

(e) If  $\neg A \leq_l A$ ,  $A \leq_l \neg A$  and  $\neg B <_l B$ , then  $A <_l B$ .

Proof: Let  $\neg A \leq_l A$ ,  $A \leq_l \neg A$  and  $\neg B <_l B$ , and suppose for reductio that  $B \leq_l A$ . Then we have the chain  $\neg B <_l B \leq_l A \leq_l \neg A$ , and thus by transitivity  $\neg B <_l \neg A$ . So by contraposition,  $A <_l B$ , and we have a contradiction. QED

**Given transitivity, connectivity, dominance and contraposition, Rabinowicz' original conjunction axiom (L4<sup>R</sup>) is equivalent with the conjunction axiom (L4) used in this paper**



(L4<sup>R</sup>) If  $\neg C <_l C$ ,  $C \leq_l A$  and  $C \leq_l B$ , then  $C \leq_l A \wedge B$

(L4) If  $\neg A <_l A$  and  $\neg B <_l B$ , then  $A \leq_l A \wedge B$  or  $B \leq_l A \wedge B$

Proof. (L4<sup>R</sup>) implies (L4). Let  $\neg A <_l A$  and  $\neg B <_l B$ . By connectivity either  $A \leq_l B$  or  $B \leq_l A$ . Suppose without loss of generality that  $A \leq_l B$  (the case  $B \leq_l A$  is similar). Since we have  $\neg A <_l A$ ,  $A \leq_l A$  as well as  $A \leq_l B$ , we can apply (L4<sup>R</sup>) and conclude that  $A \leq_l A \wedge B$ .

(L4) implies (L4<sup>R</sup>). Let  $\neg C <_l C$ ,  $C \leq_l A$  and  $C \leq_l B$ . By contraposition, we get  $\neg A \leq_l \neg C$  and  $\neg B \leq_l \neg C$ . So by several applications of transitivity,  $\neg A <_l A$  and  $\neg B <_l B$ . So by (L4), we get  $A \leq_l A \wedge B$  or  $B \leq_l A \wedge B$ . In either case, an application of transitivity gives  $C \leq_l A \wedge B$ . QED

**Given transitivity, connectivity, dominance and contraposition, the conjunction axiom (L4) can be strengthened to**

(L4<sup>+</sup>) If  $\neg A \leq_l A$  and  $\neg B <_l B$ , then  $A \leq A \wedge B$  or  $B \leq_l A \wedge B$

Proof. Let  $\neg A \leq_l A$  and  $\neg B <_l B$ . The case  $\neg A <_l A$  is covered by (L4). So assume that  $A \leq_l \neg A$ .

Suppose that not  $A \leq_l A \wedge B$ . By connectivity,  $A \wedge B <_l A$ . Thus by contraposition,  $\neg A <_l \neg(A \wedge B)$ . Using  $A \wedge B \vdash A$ , dominance,  $A \leq_l \neg A$  and transitivity, we get  $A \wedge B <_l \neg(A \wedge B)$ .

Now we can apply (L4) and get

$$B \leq_l B \wedge \neg(A \wedge B) \text{ or } \neg(A \wedge B) \leq_l B \wedge \neg(A \wedge B)$$

which can be simplified to

$$B \leq_l \neg A \wedge B \text{ or } \neg(A \wedge B) \leq_l \neg A \wedge B.$$

This implies, by dominance and transitivity

$$B \leq_l \neg A \text{ or } \neg(A \wedge B) \leq_l \neg A.$$

But the latter contradicts  $\neg A <_l \neg(A \wedge B)$  what we had above. So  $B \leq_l \neg A$  must be true. Since  $\neg A \leq_l A$  by transitivity  $B \leq_l A$ .

On the other hand, lemma (e) above tells us that  $A \leq_l \neg A$ ,  $\neg A \leq_l A$  and  $\neg B <_l B$  taken together imply that  $A <_l B$ , and we also get a contradiction.

So the supposition that not  $A \leq_l A \wedge B$  has led us into a contradiction. Thus  $A \leq_l A \wedge B$ , and we are done. QED

**The likelihood axioms (L1) – (L5) imply the following “cross-categorical” functionalities**

(LC $\wedge$ ) If  $B \leq_l \neg B <_l A$ , then  $B \leq_l A \wedge B$

(*A* belief, *B* non-belief or disbelief)

(LC $\vee$ ) If  $B <_l \neg A \leq_l A$ , then  $A \vee B \leq_l A$

(*A* belief or non-belief, *B* disbelief)

Proof. (LC $\wedge$ ) Let  $B \leq_l \neg B <_l A$ . Hence, by lemma (d),  $\neg A <_l A$ . We want to show that  $B \leq_l A \wedge B$ . By contraposition (L5), this means that  $\neg(A \wedge B) \leq_l \neg B$ . Suppose for reductio that this was not true, i.e., by connectivity (L2), that  $\neg B <_l \neg(A \wedge B)$ . Hence, by lemma (d),  $A \wedge B <_l \neg(A \wedge B)$ . Then by restricted conjunctiveness (L4), either  $A \leq_l A \wedge \neg(A \wedge B)$  or  $\neg(A \wedge B) \leq_l A \wedge \neg(A \wedge B)$ . Either way, we get by transitivity (L1) that  $\neg B <_l A \wedge \neg(A \wedge B)$ . But since  $A \wedge \neg(A \wedge B)$  implies  $\neg B$ , this contradicts dominance (L3).

(LC $\vee$ ) Let  $B <_l \neg A \leq_l A$ . Then by contraposition (L5),  $\neg A \leq_l A <_l \neg B$ . We want to show that  $A \vee B \leq_l A$ , that is, by contraposition again,  $\neg A \leq_l \neg A \wedge \neg B$ . But this follows immediately by (LC $\wedge$ ) that we have just proved. QED

**Observation 6.** Take an entrenchment relation  $\leq_e$  and the corresponding plausibility relation  $\leq_p$  satisfying the fitting condition  $A \leq_p B$  iff  $\neg B \leq_e \neg A$ . Then the relation  $\leq_l$  defined by

$$A \leq_l B \quad \text{iff} \quad \text{both } A \leq_e B \text{ and } A \leq_p B$$

is a likelihood relation.

Proof. This result is due to Rabinowicz (1995). Because of some differences in the details, we give a proof of our own.

Transitivity and dominance for  $\leq_l$ , (L1) and (L3), follow immediately from Transitivity and dominance for  $\leq_e$  and  $\leq_p$ . Contraposition (L5) follows immediately from the fitting condition.

Connectivity (L2). Suppose that not  $A \leq_l B$ . We need to show that  $B \leq_l A$ . That not  $A \leq_l B$  means that either not  $A \leq_e B$  or not  $A \leq_p B$ .

Case 1: Not  $A \leq_e B$ . Hence, by the connectivity of  $\leq_e$ ,  $B <_e A$ . By the fitting condition,  $\neg A <_p \neg B$ . Hence, by dominance and transitivity,  $\neg A <_p \top$ . Hence, since for every proposition, either it or its negation is as plausible as  $\top$ ,  $\top \leq_p A$ . Hence, by dominance and transitivity again,  $B \leq_p A$ . Taking this together with  $B <_e A$ , we get that  $B \leq_l A$ .

Case 2: Not  $A \leq_p B$ . Hence, by the connectivity of  $\leq_p$ ,  $B <_p A$ . By the fitting condition,  $\neg A <_e \neg B$ . Hence, by dominance and transitivity,  $\perp <_e \neg B$ . Hence, since for every proposition, either it or its negation is as entrenched as  $\perp$ ,  $B \leq_e \perp$ . Hence, by dominance and transitivity again,  $B \leq_e A$ . Taking this together with  $B <_p A$ , we get that  $B \leq_l A$ .

Positive conjunctiveness (L4). Let  $\neg A <_l A$  and  $\neg B <_l B$ . We need to show that either  $A \leq_l A \wedge B$  or  $B \leq_l A \wedge B$ .

By the fitting condition,  $\neg A \leq_e A$  is equivalent with  $\neg A \leq_p A$ . Thus  $\neg A <_l A$  reduces to  $\neg A <_e A$ , and similarly  $\neg B <_l B$  reduces to  $\neg B <_e B$ .

By conjunctiveness for  $\leq_e$ , we know that either  $A \leq_e A \wedge B$  or  $B \leq_e A \wedge B$ . To prove our claim, it thus suffices to show that both  $A \leq_p A \wedge B$  and  $B \leq_p A \wedge B$ , i.e., by the fitting condition, that both  $\neg(A \wedge B) \leq_e \neg A$  and  $\neg(A \wedge B) \leq_e \neg B$ . But since either  $A \leq_e A \wedge B$  or  $B \leq_e A \wedge B$ , and since both  $\perp \leq_e \neg A <_e A$  and  $\perp \leq_e \neg B <_e B$ , we know that  $\perp <_e A \wedge B$ . Hence, since for every proposition, either it or its negation is as entrenched as  $\perp$ ,  $\neg(A \wedge B) \leq_e \perp$ . Thus, by dominance and transitivity,  $\neg(A \wedge B) \leq_e \neg A$  and  $\neg(A \wedge B) \leq_e \neg B$ , which finishes the proof. QED

**It follows from (Bi)–(Biv) that if  $A \vdash B$  then  $\beta(A) \leq \beta(B)$ .**

Proof. Suppose that  $A \vdash B$ . Since the logic is classical, this is equivalent with each of the following conditions:  $\neg B \vdash \neg A$ ,  $A \dashv\vdash A \wedge B$ ,  $\vdash \neg A \vee B$  and  $A \wedge \neg B \vdash \perp$ . We want to show that  $\beta(A) \leq \beta(B)$ . In order to do this, we distinguish six cases.

Case 1.  $\beta(A) > 0$  and  $\beta(B) > 0$ . Then, by (Biii),  $\beta(A \wedge B) = \min\{\beta(A), \beta(B)\}$ . By (Bi),  $\beta(A) = \beta(A \wedge B)$ , so  $\beta(A) \leq \beta(B)$ , as desired.

Case 2.  $\beta(A) > 0$  and  $\beta(B) < 0$ . By inversion,  $\beta(\neg B) > 0$ . So, by (Biii),  $\beta(A \wedge \neg B) = \min\{\beta(A), \beta(\neg B)\} > 0$ . By (Bi),  $\beta(A \wedge \neg B) = \beta(\perp)$ , so  $\beta(\perp) > 0$ . By inversion again,  $\beta(\top) < 0$ , contradicting (Bii). So this case is impossible.

Case 3.  $\beta(A) > 0$  and  $\beta(B) = 0$ . By inversion,  $\beta(\neg B) = 0$ . Thus, by (Biii),  $\beta(A \wedge \neg B) = \min\{\beta(A), \beta(\neg B)\} = 0$ . By (Bi),  $\beta(A \wedge \neg B) = \beta(\perp)$ , so  $\beta(\perp) = 0$  and, by inversion,  $\beta(\top) = 0$ . We now show that for all sentences  $C$ ,  $\beta(C) = 0$ . Suppose firstly for reductio that  $\beta(C) > 0$ . Then  $\beta(C) = \beta(C \wedge \top) = \min\{\beta(C), \beta(\top)\} = 0$ , and we have a contradiction. Suppose secondly for reductio that  $\beta(C) < 0$ . By inversion,  $\beta(\neg C) > 0$ . Then  $\beta(\neg C) = \beta(\neg C \wedge \top) = \min\{\beta(\neg C), \beta(\top)\} = 0$ , and we have again a contradiction. Thus  $\beta$  is the constant function assigning 0 to all sentences, so trivially  $\beta(A) \leq \beta(B)$ . (It is in fact immediate that the relational projection of this function satisfies all of (L1)–(L5).)

Case 4.  $\beta(A) \leq 0$  and  $\beta(B) \geq 0$ . This immediately implies  $\beta(A) \leq \beta(B)$ .

Case 5.  $\beta(A) < 0$  and  $\beta(B) < 0$ . Then, by inversion (Biv),  $\beta(\neg A) > 0$  and  $\beta(\neg B) > 0$ . By (Biii),  $\beta(\neg A \wedge \neg B) = \min\{\beta(\neg A), \beta(\neg B)\}$ . By (Bi),  $\beta(\neg B) = \beta(\neg A \wedge \neg B)$ , so  $\beta(\neg B) \leq \beta(\neg A)$ . Thus, by inversion (Biv) again,  $\beta(A) \leq \beta(B)$ , as desired.

Case 6.  $\beta(A) = 0$  and  $\beta(B) < 0$ . By inversion,  $\beta(\neg B) > 0$ . Thus, by (Biii),  $\beta(A \wedge \neg B) = \min\{\beta(A), \beta(\neg B)\} = 0$ , and the case continues exactly like case 2. QED

**Observation 7. A function  $\beta$  is a likelihood function satisfying (Bi)–**

**(Biv) if and only if there is an entrenchment function  $\varepsilon$  satisfying (Ei)–(Eiii) such that**

$$\beta(A) = \varepsilon(A) + \pi(A)$$

**where  $\pi$  is the plausibility function corresponding to  $\varepsilon$ , defined by  $\pi(A) = -\varepsilon(\neg A)$ .**

Proof. We first show that for every entrenchment function  $\varepsilon$  satisfying (Ei)–(Eiii), the function  $\beta$  defined by  $\beta(A) = \varepsilon(A) - \varepsilon(\neg A)$  is a likelihood function satisfying (Bi)–(Biv). Intensionality (Bi) and inversion (Biv) follow immediately from the intensionality of  $\varepsilon$  and the definition of  $\beta$ . For (Bii),  $\beta(\top) = \varepsilon(\top) - \varepsilon(\perp) \geq 0$ , since  $\varepsilon$  is non-negative and  $\varepsilon(\perp) = 0$ , by (Eii). The most complex condition is (Biii). Suppose that  $\beta(A) \geq 0$  and  $\beta(B) > 0$ . We need to show that  $\beta(A \wedge B) = \min\{\beta(A), \beta(B)\}$ . From  $\beta(A) \geq 0$  we conclude that  $\varepsilon(\neg A) = 0$  and  $\beta(A) = \varepsilon(A)$ . From  $\beta(B) > 0$  we conclude that  $\varepsilon(\neg B) = 0$  and  $\beta(B) = \varepsilon(B)$ . From  $\varepsilon(\neg A) = 0$  and  $\varepsilon(B) > 0$ , we get that  $\varepsilon(\neg(A \wedge B)) = 0$ , otherwise  $\varepsilon(\neg A) \geq \varepsilon(B \wedge \neg(A \wedge B)) = \min\{\varepsilon(B), \varepsilon(\neg(A \wedge B))\} > 0$ . Now finally consider  $\beta(A \wedge B) = \varepsilon(A \wedge B) - \varepsilon(\neg(A \wedge B))$ . Since  $\varepsilon(A \wedge B) = \min\{\varepsilon(A), \varepsilon(B)\}$  and  $\varepsilon(\neg(A \wedge B)) = 0$ , we get that  $\beta(A \wedge B) = \min\{\beta(A), \beta(B)\}$ , as desired.

For the converse direction, let  $\beta$  be a likelihood function satisfying (Bi)–(Biv). We define for all sentences  $A$

$$\varepsilon(A) = \max\{\beta(A), 0\}$$

with the corresponding plausibility function being

$$\pi(A) = -\varepsilon(\neg A) = -\max\{\beta(\neg A), 0\} = -\max\{-\beta(A), 0\} = \min\{\beta(A), 0\}$$

Now we check that this function  $\varepsilon$  indeed generates  $\beta$  by means of the equation  $\beta(A) = \varepsilon(A) - \varepsilon(\neg A)$ . Let us distinguish two cases. If  $\beta(A) \geq 0$ , then by inversion  $\beta(\neg A) \leq 0$ , so  $\varepsilon(A) - \varepsilon(\neg A) = \max\{\beta(A), 0\} - \max\{\beta(\neg A), 0\} = \beta(A)$ . If  $\beta(A) < 0$ , then by inversion  $\beta(\neg A) > 0$ , so  $\varepsilon(A) - \varepsilon(\neg A) = \max\{\beta(A), 0\} - \max\{\beta(\neg A), 0\} = -\max\{-\beta(A), 0\} = \beta(A)$ . So in either case  $\beta(A) = \varepsilon(A) - \varepsilon(\neg A)$ , as desired.

We finally show that  $\varepsilon$  is an entrenchment function satisfying (Ei)–(Eiii). (Ei) and (Eii) are immediate. Regarding (Eiii), the case when both  $\varepsilon(A)$  and  $\varepsilon(B)$  are positive is directly covered by the positive conjunctiveness condition (Biii). The only subcase of (Eiii) that requires a closer look is when either  $\varepsilon(A) = 0$  or  $\varepsilon(B) = 0$ . So suppose without loss of generality that  $\varepsilon(A) = 0$ . Then, by the definition of  $\varepsilon$ ,  $\beta(A) \leq 0$ . We need to show that  $\varepsilon(A \wedge B) = \min\{\varepsilon(A), \varepsilon(B)\} = 0$ , i.e., that  $\beta(A \wedge B) \leq 0$ . Suppose for reductio that  $\beta(A \wedge B) > 0$ . Then by dominance for  $\beta$  which we have proved to be satisfied before,  $\beta(A) > 0$ , too, and we have a contradiction. QED

**Observation 8.** Take a Spohnian beta function  $\beta$ . Then its relational projection defined by

$$A \leq_l B \text{ iff } \beta(A) \leq \beta(B)$$

is a Rabinowicz likelihood relation.

Conversely, for every finite Rabinowicz likelihood relation  $\leq_l$  there is a Spohnian beta function  $\beta$  such that  $\leq_l$  is the relational projection of  $\beta$ .

Proof. Part 1. That the relational projection  $\leq_l$  of a beta function satisfies (L1), (L2), (L4) and (L5) follows trivially from the conditions (Bi)–(Biv). And we have proven before that these conditions imply a dominance condition for  $\beta$  which in turn guarantees the dominance condition (L3) for  $\leq_l$ .

Part 2 is sketched in the main text, after the formulation of Obs. 8. QED

**The relation  $\leq_{ee} = \leq_e \cap \leq_{ex}$  is a comparative necessity relation satisfying (E1)–(E3)**

Proof.  $\leq_{ee}$  satisfies transitivity and dominance, since both  $\leq_e$  and  $\leq_{ex}$  do.

For conjunctiveness, suppose that not  $A \leq_{ee} A \wedge B$ . We need to show that  $B \leq_{ee} A \wedge B$ . That not  $A \leq_{ee} A \wedge B$  can come about in two ways: Either not  $A \leq_e A \wedge B$  or not  $A \leq_{ex} A \wedge B$ .

Case 1. Suppose that not  $A \leq_e A \wedge B$ . So  $B \leq_e A \wedge B$ , by the conjunctiveness of  $\leq_e$ . Also,  $A \wedge B <_e A$ , by the connectivity of  $\leq_e$ , and thus a fortiori  $\perp <_e A$ . By the fitting condition (i), we get  $\top \leq_{ex} A$  and thus a fortiori  $B \leq_{ex} A$ . So by the conjunctiveness and transitivity of  $\leq_{ex}$ ,  $B \leq_{ex} A \wedge B$ . Since we also had  $B \leq_e A \wedge B$ , we finally get  $B \leq_{ee} A \wedge B$ , as desired.

Case 2. Suppose that not  $A \leq_{ex} A \wedge B$ . So  $B \leq_{ex} A \wedge B$ , by the conjunctiveness of  $\leq_{ex}$ . Also,  $A \wedge B <_{ex} A$ , by the connectivity of  $\leq_{ex}$ . By the transitivity of  $\leq_{ex}$ , we get  $B <_{ex} A$ , and thus a fortiori  $B <_{ex} \top$ . By the fitting condition (i), we get  $B \leq_e \perp$  and thus a fortiori  $B \leq_e A \wedge B$ . Since we also had  $B \leq_{ex} A \wedge B$ , we finally get  $B \leq_{ee} A \wedge B$ , as desired. QED

**The relation  $\leq_{pp} = \leq_p \cap \leq_{px}$  is a comparative possibility relation satisfying (P1)–(P3)**

Proof.  $\leq_{pp}$  satisfies transitivity and dominance, since both  $\leq_p$  and  $\leq_{px}$  do.

For disjunctiveness, suppose that not  $A \vee B \leq_{pp} A$ . We need to show that  $A \vee B \leq_{pp} B$ . That not  $A \vee B \leq_{pp} A$  can come about in two ways: Either not  $A \vee B \leq_p A$  or not  $A \vee B \leq_{px} A$ .

Case 1. Suppose that not  $A \vee B \leq_p A$ . So  $A \vee B \leq_p B$ , by the disjunctiveness of  $\leq_p$ . Also,  $A <_p A \vee B$ , by the connectivity of  $\leq_p$ , and thus a fortiori  $A <_p \top$ .

By the fitting condition (ii), we get  $A \leq_{px} \perp$  and thus a fortiori  $A \leq_{px} B$ . So by the disjunctiveness and transitivity of  $\leq_{px}$ ,  $A \vee B \leq_{px} B$ . Since we also had  $A \vee B \leq_p B$ , we finally get  $A \vee B \leq_{pp} B$ , as desired.

Case 2. Suppose that not  $A \vee B \leq_{px} A$ . So  $A \vee B \leq_{px} B$ , by the disjunctiveness of  $\leq_{px}$ . Also,  $A <_{px} A \vee B$ , by the connectivity of  $\leq_{px}$ . By the transitivity of  $\leq_{px}$ , we get  $A <_{px} B$ , and thus a fortiori  $\perp <_{px} B$ . By the fitting condition (ii), we get  $\top \leq_p B$  and thus a fortiori  $A \vee B \leq_p B$ . Since we also had  $A \vee B \leq_{px} B$ , we finally get  $A \vee B \leq_{pp} B$ , as desired. QED

**A lemma concerning the relation  $\leq_{ll} = \leq_l \cap \leq_{lx}$ , built from  $\leq_l$  and  $\leq_{lx}$ . Let  $\leq_l$  and  $\leq_{lx}$  satisfy the fitting condition (iii). Then**

- (i)      **If  $A <_l B$ , then  $A \leq_{lx} B$**
- (ii)     **If  $A <_{lx} B$ , then  $A \leq_l B$**
- (iii)    **If  $A \leq_{lx} \neg A$ , then  $A \leq_l \neg A$**
- (iv)      **$\neg A <_{ll} A$  if and only if  $\neg A <_{lx} A$**

Proof. (i) Let  $A <_l B$  and suppose for reductio that not  $A \leq_{lx} B$ . From the latter we get by connectivity that  $B <_{lx} A$ , and a fortiori  $B <_{lx} \top$ . Hence, by the fitting condition (iii),  $B \leq_l \neg B$ . Taken together with  $A <_l B$ , this gives us  $A <_l \neg B$ . By contraposition,  $B <_l \neg A$ . So by transitivity,  $A <_l \neg A$ . By the fitting condition (iii) again, we get  $\top \leq_{lx} \neg A$ , and a fortiori  $\neg B \leq_{lx} \neg A$ . By contraposition,  $A \leq_{lx} B$ , and we have a contradiction.

(ii) Let  $A <_{lx} B$  and suppose for reductio that not  $A \leq_l B$ , that is,  $B <_l A$ . From the former we get a fortiori  $A <_{lx} \top$ , so by the fitting condition (iii),  $A \leq_l \neg A$ . By transitivity, we get  $B <_l \neg A$ , and by contraposition  $A <_l \neg B$ . By transitivity again, this gives us  $B <_l \neg B$ , so by the fitting condition (iii) again,  $\top \leq_{lx} \neg B$ . So a fortiori,  $\neg A \leq_{lx} \neg B$ , and by contraposition  $B \leq_{lx} A$ . But this contradicts the initial supposition  $A <_{lx} B$ .

(iii) Let  $A \leq_{lx} \neg A$ , and suppose for reductio that not  $A \leq_l \neg A$ , that is,  $\neg A <_l A$ . From the latter we get, by the fitting condition (iii),  $\top \leq_{lx} A$ . By transitivity, this gives us  $\top \leq_{lx} \neg A$ , and by the fitting condition (iii) again,  $A <_l \neg A$ , and we have a contradiction. We conclude that  $A \leq_l \neg A$ , as desired.

(iv) The condition  $\neg A <_{ll} A$  means that  $\neg A \leq_{ll} A$  and not  $A \leq_{ll} \neg A$ . The former part says that

$$\neg A \leq_l A \quad \text{and} \quad \neg A \leq_{lx} A$$

while the latter part says that

$$\text{not } A \leq_l \neg A \quad \text{or} \quad \text{not } A \leq_{lx} \neg A$$

By part (iii) of this lemma, the latter line means that not  $A \leq_{lx} \neg A$ , i.e., by connectivity  $\neg A <_{lx} A$ . But, due to part (iii) of this lemma again, this implies

the former part. In sum then,  $\neg A <_{ll} A$  is equivalent with  $\neg A <_{lx} A$ . QED

**The relation  $\leq_{ll} = \leq_l \cap \leq_{lx}$  is a likelihood relation satisfying (L1)–(L5)**

Proof.  $\leq_{ll}$  satisfies transitivity (L1), dominance (L3) and contraposition (L5), since both  $\leq_l$  and  $\leq_{lx}$  do.

For connectivity (L2), assume that not  $A \leq_{ll} B$ . We need to show that  $B \leq_{ll} A$ , that is,  $B \leq_l A$  and  $B \leq_{lx} A$ . That not  $A \leq_{ll} B$  can come about in two ways: Either not  $A \leq_l B$  or not  $A \leq_{lx} B$ . Firstly, suppose that not  $A \leq_l B$ . Then, by connectivity,  $B <_l A$ . By part (i) of the lemma, we also get  $B \leq_{lx} A$ , and we are done. Suppose secondly that not  $A \leq_{lx} B$ . Then, by connectivity,  $B <_{lx} A$ . By part (ii) of the lemma, we also get  $B \leq_l A$ , and we are done.

For positive conjunctiveness (L4), let  $\neg A <_{ll} A$  and  $\neg B <_{ll} B$ . By the lemma, part (iv), this reduces to  $\neg A <_{lx} A$  and  $\neg B <_{lx} B$ . Assume further that not  $A \leq_{ll} A \wedge B$ . We need to show that  $B \leq_{ll} A \wedge B$ , that is  $B \leq_l A \wedge B$  and  $B \leq_{lx} A \wedge B$ . That not  $A \leq_{ll} A \wedge B$  can come about in two ways: Either not  $A \leq_l A \wedge B$  or not  $A \leq_{lx} A \wedge B$ .

Case 1. Suppose that not  $A \leq_l A \wedge B$ . Then, by connectivity,  $A \wedge B <_l A$ . Suppose for reductio that not  $B \leq_{lx} A \wedge B$ , that is  $A \wedge B <_{lx} B$ . Then by positive conjunctiveness for  $\leq_{lx}$ ,  $A \leq_{lx} A \wedge B$ . By transitivity,  $A <_{lx} B$ . Thus, by part (ii) of the lemma,  $A \leq_l B$ . Together with  $A \wedge B <_l A$ , this gives us  $A \wedge B <_l B$ , by transitivity. We conclude with positive conjunctiveness for  $\leq_l$  that either  $A \leq_l \neg A$  or  $B \leq_l \neg B$ . By  $A \leq_l B$ , contraposition and transitivity, this reduces to  $A \leq_l \neg A$ . Together with  $A \wedge B <_l A$ , we get  $A \wedge B <_l \neg A$ . However, we have  $\neg A <_{lx} A$  as well as  $A \leq_{lx} A \wedge B$  which implies  $\neg A <_{lx} A \wedge B$ . By the lemma, part (ii), this implies  $\neg A \leq_l A \wedge B$ , and we have a contraction. So the supposition was wrong, and we have shown that  $B \leq_{lx} A \wedge B$ , as desired.

Case 2. Suppose that not  $A \leq_{lx} A \wedge B$ . Then, by connectivity,  $A \wedge B <_{lx} A$ , and by positive conjunctiveness for  $\leq_{lx}$ ,  $B \leq_{lx} A \wedge B$ . Taken together, this gives us  $B <_{lx} A$ , and by part (ii) of the lemma  $B \leq_l A$ . Suppose for reductio that not  $B \leq_l A \wedge B$ , that is  $A \wedge B <_l B$ . By transitivity, we get  $A \wedge B <_l A$ . By positive conjunctiveness of  $\leq_l$ , we conclude that either  $A \leq_l \neg A$  or  $B \leq_l \neg B$ . By  $B \leq_l A$ , contraposition and transitivity, this reduces to  $B \leq_l \neg B$ . Taken together with  $A \wedge B <_l B$ , this gives us that  $A \wedge B <_l \neg B$ . However, we have  $\neg B <_{lx} B$  as well as  $B \leq_{lx} A \wedge B$ , which implies  $\neg B <_{lx} A \wedge B$ . By the lemma, part (ii), this implies  $\neg B \leq_l A \wedge B$ , and we have a contradiction. So the supposition was wrong, and we have shown that  $B \leq_l A \wedge B$ , as desired. QED

## APPENDIX II: THE MODAL LOGIC OF PLAIN BELIEF AS IMPLICIT IN THE LOGIC OF ENTRENCHMENT RELATIONS

According to the Lockean thesis, a proposition can count as believed if the degree of confidence in its truth is sufficiently high, or in the terms mainly used in this paper, if the degree of belief in it is high enough. Degrees of belief are here thought of as comparative necessity relations, also called ‘entrenchment relations’. What kind of implications does the theory of entrenchment relations and functions developed in this paper have for a logic of plain belief? In order to answer this question we have to begin with another one: How can we translate statements of comparative necessity into the language of plain belief?

The Lockean thesis implies that belief is upward-closed, that is, if  $A$  is believed and  $A \leq B$ , then  $B$  is believed as well. In a first approximation, let us thus read  $A \leq B$  as expressing ‘*If the reasoner believes  $A$ , then she also believes  $B$* ’, or more formally,  $\Box A \rightarrow \Box B$ . For this suggestion to make sense, we have to presuppose the transitivity condition (E1) for  $\leq$ .

We then have interesting translations of the entrenchment axioms. For the labelling and the systematic place of the respective axioms in modal logic, see Chellas (1980, Chapter 8).<sup>38</sup> Dominance (E2) becomes Chellas’ rule

(RM) From  $\vdash A \rightarrow B$  infer  $\vdash \Box A \rightarrow \Box B$

Conjunctiveness (E3) becomes Chellas’ axiom

(C)  $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$

Taken together, (RM) and (C) define a *regular* system of modal logic (Chellas’ terminology). Alternatively, in the place of (RM) one could use the weaker rule

(RE) From  $\vdash A \leftrightarrow B$  infer  $\vdash \Box A \leftrightarrow \Box B$

together with the additional axiom

(M)  $\Box(A \wedge B) \rightarrow \Box A \wedge \Box B$

Regular systems are still weaker than *normal* systems.<sup>39</sup> I think it may perhaps be said that (C) and (M) provide a syntactic way of capturing the intrinsic meaning of ‘ $\Box$ ’. It is characteristic of the concept of necessity that it distributes over conjunction.

What is still missing is the necessitation rule (RN) ‘From  $\vdash A$  infer  $\vdash \Box A$ ’, or equivalently, the axiom (N),  $\Box \top$ . Systems without (RN) or (N) have no theorems of the form ‘ $\Box A$ ’, so no beliefs at all are declared by them as “logically required”.

It is hard to express this in entrenchment language. Relatively close is  $\perp < \top$ . Clearly  $\perp \leq \top$ , saying that  $\Box \perp \rightarrow \Box \top$ , is just an instance of (E2). In our first

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<sup>38</sup>Here as everywhere in this paper, I use the term ‘axiom’ also when I talk about axiom schemes.

<sup>39</sup>Instead of (M), just  $\Box(A \wedge B) \rightarrow \Box A$  would be sufficient. Notice that (RE) corresponds to (Ei), and the biconditional (R) joining (C) and (M) corresponds to (Eiii).



approximation to the interpretation of  $\leq$ , ‘Not  $\top \leq \perp$ ’ is just construed as the negated material conditional ‘ $\neg(\Box\top \rightarrow \Box\perp)$ ’. Thus the condition  $\perp < \top$  seems to express, roughly, that all tautologies, but no contradictions are to be believed, and it makes sense to stipulate this. It is equivalent with the conjunction of the modal axiom<sup>40</sup>

$$(N) \quad \Box\top$$

with the consistency axiom

$$(P) \quad \neg\Box\perp$$

(P) is equivalent, in the context of the other axioms, with the usual axiom stating that whatever is necessary is possible

$$(D) \quad \Box A \rightarrow \Diamond A$$

Using the interdefinability of  $\Box$  and  $\Diamond$  as an axiom, it is easy to show that the following rule and axioms are the counterparts of (RE), (M) and (C) for the possibility operator:

$$(RE\Diamond) \quad \text{From } \vdash A \leftrightarrow B \text{ infer } \vdash \Diamond A \leftrightarrow \Diamond B$$

$$(M\Diamond) \quad (\Diamond A \vee \Diamond B) \rightarrow \Diamond(A \vee B)$$

$$(C\Diamond) \quad \Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$$

We now understand how fundamental the role of the Distribution Laws for  $\Box$  and  $\Diamond$  is for the characterization of necessity and possibility, respectively. Given (RE) or (RE $\Diamond$ ), they suffice to characterize regular systems of modal logic.

Let us summarize the situation as seen in the light of our first approximation to the modal reading of entrenchments (i.e., of comparative necessities). The modal logic of belief implicit in our standard axiomatizations (E1)–(E3) or (Ei)–(Eiii), is the non-normal, *regular* system of modal logic called *R* or *ECM* by Chellas; an axiomatization in the spirit of this paper consists in (RE), (C) and (M). If we decide to add the non-triviality condition  $\perp < \top$  or  $\varepsilon(\perp) < \varepsilon(\top)$ , then we get a double extension of *ECM*: The normal system *D* or *KD* satisfying the axioms (N) and (D). Entrenchments do not validate the truth axiom (T), nor do they account for iterated modalities.

But there is a problem about our first approximation. This can be seen in the translation of the connectivity condition for  $\leq$ , for instance. We should not render it by  $(\Box A \rightarrow \Box B) \vee (\Box B \rightarrow \Box A)$  which would be a simple tautology of propositional logic. Connectivity should not be that trivial. So I don’t think ‘Not  $A \leq B$ ’ should be read as  $\neg(\Box A \rightarrow \Box B)$ , i.e.,  $\Box A \wedge \neg\Box B$ , because this says that *A* is *actually* believed and *B* is *actually* not believed. Now what else could it mean? The correct meaning is rather that *there is a level of belief* at which *A* is believed but *B* is not believed. This does not imply that the level referred to is the one actually applied by, or actually ascribed to, the reasoner for demarcating her ‘beliefs’ from her ‘non-beliefs’.

<sup>40</sup>About which Chellas does not say much.

Better than the first approximation, and I think basically correct, is it to read  $A \leq B$  as expressing ‘*Whenever the reasoner believes A, then she also believes B*’. The quantification is not over time indices here, but over degrees or levels of belief. We can be more precise about that if the language includes (finitely many) graded modalities  $\Box_1, \dots, \Box_n$  and their respective duals  $\Diamond_1, \dots, \Diamond_n$ , governed by the logical axioms  $\Box_i A \rightarrow \Box_j A$  or, respectively,  $\Diamond_j A \rightarrow \Diamond_i A$  for all  $i$  and  $j$  such that  $i \geq j$  (cf. Goble 1970). Then we can say that  $A \leq B$  means that for all certainty indices  $i$ , ‘If the reasoner believes  $A$  at level  $i$ , then she also believes  $B$  at  $i$ ’, or more formally, for all  $i$ ,  $\Box_i A \rightarrow \Box_i B$ . Connectivity is no longer trivial on this reading. Having relativized belief to some grade or level, however, we find it hard to make sense of the notion of plain belief.